APPLICATION OF MATHEMATICAL PROGRAMMING

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IN

OPERATIONS RESEARCH

By

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PREFACE

This Dissertation entitled "***Application of Mathematical programming***" is submitted to the Aligarh Muslim University, Aligarh, for the partial fulfillment of the degree of M.Phil in OperationsResearch.

Mathematical programming is concerned with the determination of a minimum or a maximum of a function of several variables, which are required to satisfy a number of constraints. Such solutions are sought in diverse fields; including engineering, Operations Research, Management Science, numerical analysis and economics etc.

This manuscript consists of five chapters. Chapter-1 deals with the brief history of Mathematical Programming. It also contains numerous applications of Mathematical Programming.

Chapter-2 gives an introduction of Transportation problem and devoted to extensions and methods of solutions. A numerical example and Relation of Transportation problems with Network problems are also discussed.

Chapter – 3 gives an introduction of Game theory. The solution procedure of Game by linear programming approach with an example has been discussed. It also contains extensions and methods of solutions of games.

Chapter- 4 deals with the application of Mathematical programming to the Cargo-Loading Problem and devoted to methods of solutions with examples.

In the last chapter, A Non-linear integer programming formulation of optimum allocation in stratified random sampling is discussed. Also its solution procedure and a variation of the problem is given.

I have great pleasure to avail this opportunity to acknowledge my indebtedness to my supervisor, Dr. A. Bari, under whose invaluable guidance and constant encouragement, this work has been completed.

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14.5.02

TEG ALAM

CONTENTS

Chapter-1

INTRODUCTION

1.1 BRIEF HISTORY; DEVELOPMENT OF MATHEMATICAL PROGRAMMING

Mathematical programming is a relatively young field, having emerged during the middle decades of the twentieth century. It has become a standard tool for improving the efficiency of business operations around the world. Indeed, a significant part of the continuing improvements in the economic productivity of the industrialized nations can be attributed to the use of mathematical programming. Since the end of world war II, mathematical programming developed rapidly as a new field of study dealing with applications of the scientific method to business operations and management decision making. But we can trace the existence of optimization methods to the days of Newton, Lagrange and Cauchy. The differential calculus methods of optimization was introduced by Newton and Leibnitz. The foundation of calculus of variations was laid by Bernoulli, Euler, Lagrange and Weirstres. Lagrange introduced his famous Lagrange Multiplier Technique to solve the constrained optimization problems. Cauchy made the first application of the steepest Decent method to solve unconstrained minimization problems. In spite of these early contributions, very little progress was made until the middle of the twentieth century.

The Linear Programming problem was first conceived by George B. Dantzig, around 1947, while he was working as a mathematical adviser to the United States Air Force comptroller on developing a mechanized planning tool for a time staged deployment, training and logistical supply program. The early applications were primarily limited to problems involving military operations, such as military logistics problems, military transportation problems, procurement problems, and other related fields. The numerical procedure for solving a linear programming problem introduced by Dantzig is known as Simplex Method. But the method was not available until is was published in the cowles Commission Monograph No. 13 in 1951.

Kuhn H.W. and Tucker A.W. (1951) published an important paper, "Nonlinear programming", dealing with necessary and sufficient conditions for optimal solutions to mathematical programming problems, which laid to the foundations for a great deal of later work in nonlinear programming.

Charnes A. and Lemke C. (1954) published an approximation method of treating problems with separable objective function subject to linear constraints. Later the technique was generalized by Miller to include separable constraints. In 1955, a number of papers by different authors dealing with the quadratic programming began to appear. Beale (1959) gave a method for quadratic programming. In the same year Wolfe p. transformed the quadratic programming problem into an equivalent linear programming problem, using K.T. conditions, which could be solved by Simplex Method. The other authors who gave techniques for solving quadratic programming problem were Houthakher H.S. (1960), Lemke C.E. (1962), Panne and Whinston (1964), Grave R.L. (1967), Alloin G. (1970) and Several others.

Interest in integer solution to linear programming problems arose early in the development of the filed. One of the first paper to be concerned with the subject was that published by Dantzig, fulkerson and Johnson in 1954. Markowitz (1957) and Manne discussed numerical techniques and some nonlinear programming problems which could be solved by integer linear programming. Ralph E. Gomory was the first to set forth a systematic computational technique for solving an integer linear programming problem for which it could be proved that convergence would be obtained in a finite number of iterations. This was done in 1958 for all integer case and in 1960 for the mixed integer case. A.H. Land and A.G. Doig (1960) gave a method, which is especially appropriate for mixed integer programming. E.l. Lawler and D.E. Wood in 1966 applied the Branch and Bound technique of Land and Doig to various non-linear programming problems like the traveling salesman problem, the quadratic assignment problem. Wei Xuan Xu in (1981) gave a new bounding technique for the quadratic assignment problem.

Richard Bellman, made the major original contribution to the development of the dynamic programming technique and published his result in about 100 papers throughout the 1950's. A summary of this work is contained in his book, "Dynamic Programming" published in 1957, and in the book, "Applied Dynamic Programming," Co-authored with S. Dreyfus and published in 1962. Dynamic programming problems paved the way for development of the methods of constrained optimization. The contributions of Zoutendijk (1966) and Rosen to nonlinear programming during the early part of the 1960's have been very significant. Although no single technique has been found to be universely applicable for nonlinear programming problems. The workd by carrol (1961) and Fiacco and McCormick (1968) made many a difficult problem to be solved by using the well known techniques of unconstrained optimization. R. Hooke and T.A. Jeeves gave a direct search method in 1961 for unconstrained optimization. M.J.d. Powell (1964) gave an efficient method for finding the minimum of a function of several variables without calculating derivatives. The other authors who made contribution for unconstrained optimsation are H.H. Rosenbroack (1960), R. Fletcher and C.M. Reeves (1964), and W.C. Davidon (1968), R. Fletcher (1970) gave a new approach to variable metric algorithms. L. Grandinetti (1982) gave an updating formula for quasi-Newton minimization algorithm.

Geometric programming was developed in the 1960's by R.J. Duffin, E. Peterson and C. Zener, Geometric Programming provides a systematic method for formulating and solving the class of optimization problems that tend to appear mainly in engineering designs. D.S. Ermer (1971) used geometric programming for optimizing the constrained machinery economics problem. Works are still going on geometric programming and its sensitivity analysis. R.S. Dembo (1982) applied sensitivity analysis in geometric programming.

G.B. Dantzig (1955), Charnes A. and W.W. Cooper (1959) developed stochastic programming techniques and solved problems by assuming design parameters to be independent and normally distributed. The basic idea used in solving any stochastic programming problem is to convert the stochastic problem into an equivalent deterministic problem. The desire to optimize more than one objective or goal while satisfying the physical limitations led to the development of multi objective programming methods. Goal programming is a well known technique for solving specific types of multi objective optimization problems. The goal programming was originally proposed for linear problems by Charnes and Cooper (1961).

Development for new techniques for solving mathematical programming are still going on, Kachian (1979) gave a polynomial algorithm for linear programming. N.

Karmarkar (1984) gave an excellent method for solving linear programming problem. His method is named as New Polynomial-Time algorithm for linear programming. A recent contribution to integer programming was due to Saltzman and Hiller (1991).

1.2 BRANCHES OF MATHEMATICAL PROGRAMMING:

The mathematical programming techniques are to find the minimum or maximum of a function of several variables under a prescribed set of constraints, thus depending on the nature of these functions and the restrictions on the decision variables; Mathematical Programming problem can be broadly classified into two categories.

(i) Linear Programming Problem (LPP)

(ii) Nonlinear programming problem (NLPP)

1.2.1 LINEAR PROGRAMMING PROBLEM:

Problems of finding non negative (≥ 0) values of a given set of variables which minimize or maximize a linear function of these variables under certain restrictions specified by linear inequalities or equations is called a linear programming problem.

The general mathematical model of on LLP is given as:

Minimize or (Maximize) $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$ .... (i)

Subject to,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \text{----------} + a_{1n}x_n \leq \text{ or } = \text{ or } \geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{----------} + a_{2n}x_n \leq \text{ or } = \text{ or } \geq b_2 \\ a_{m1}x_1 + a_{m2}x_2 + \text{-------} + a_{mn}x_n \leq \text{ or } = \text{ or } \geq b_m \end{array}\right\} \ldots\ldots\ldots(ii)$$

and $x_1, x_2, \text{-------}, x_n \geq 0$ ....(iii)

Expression (i) is called the ***objective function*** of the given LPP. Conditions in (ii) are called the ***"Constraints"*** of the given LPP & Restrictions in (iii) are called the ***"non-negatively restrictions"***.

CONVERSION OF AN LPP INTO STANDARD FORM:

Min or (Max) $Z = \sum_{j=1}^{n} c_j x_j$

Subject to, $\sum_{j=1}^{n} a_{ij} x_j \leq$ or $= \geq$ b_i; $i = 1, 2, \ldots m$

And $x_j \geq 0$, $j = 1, 2, \text{------} n$

Or

Min or (Max) $Z = \underline{c}\ \underline{x}$

Subject to, $A\underline{x} \leq$ or $=$ or $\geq$

And $\underline{x} \geq \underline{0}$

LPP in standard form,

Minimize $Z = \underline{c}'\ \underline{x}$

Subject to, $A\underline{x} = \underline{b}$

and $\underline{x} \geq 0$= requirement vector

where, $\underline{c}$ = cost vector; $\underline{x}$ = Decision Vector; and, A = Matrix of coefficient problem matrix

If the given LPP is not in the above standard form, it can be easily converted into it by simple algebric operations as given below.

(i) If the objective is to be maximized, it can be minimized after multiplying by '-1'.

(ii) If some of constraints are of ($\leq$) type, they can be converted into equations by adding a quantity $x \geq 0$ from L.H.S. of constraints, this x is called ***slack variable***.

Similarly if some of the constraints are of ($\geq$) type, they can be converted into equations by subtracting a quantity $y \geq 0$ from L.H.S. of constraints, this y is called ***surplus variable.***

(iii) If some of the variable $x_j \leq 0$, we can set, $-x_j = x'_j$ where the new variable $x'_j \geq 0$

(iv) If some of variable x_j are unrestricted in sign, they can be expressed as a difference of two (≥ 0) variables as:

$$x_j = x'_j - x''_j; \text{ where } x'_j, x''_j \geq 0$$

PROPERTIES OF L.P. PROBLEM:

Linear programming problems posses certain specific properties which make these problems easier to solve as compared to nonlinear programming problems. These properties are:

(i) The set of feasible solutions which satisfy the constraints and the non-negativity restriction is a convex set having a finite number of extreme points (corners).

(ii) The set of all vectors $\underline{x} = (x_1, x_2 \ldots\ldots\ldots x_n)$ which yield a specified value of the objective function is a hyper plane. Further more the hyper planes corresponding to different values of the objective function are parallel.

(iii) If the optimal value of the objective function is bounded, at least one of the extreme point of the convex set will be an optimal solution and starting at any extreme point, it is possible to reach an optimal extreme point in a series of steps such that at each step one moves along to an adjacent extreme point.

(iv) A given extreme point is optimal if and only if the value of the objective function at that extreme point is atleast as great as the value of the objective function at each adjacent extreme points.

In nonlinear programming problem some or all of these features may be absent creating difficulties in solving them.

1.2.2 METHODS FOR SOLUTIONS OF L.P.P. :

Linear programming is a powerful technique for dealing with the problem of allocating limited resources among competing activities as well as other problems having a similar mathematical formulation. It has become a standard tool of great importance for numerous business and industrial organizations. Furthermore, almost any social organization is concerned with allocating resources in some context, and there is a growing recognition of the extremely wide applicability of this technique. However, not all problem of allocating limited resources can be formulated to fit a linear programming model, even as a reasonable approximation. When one or more of the assumptions of linear programming is violated seriously, it may then be possible to apply another mathematical programming model instead, e.g. the models of integer programming or nonlinear programming.

Linear programming and the simplex method were developed by Dantzig in 1947 in connection with military planning. For discussion on block pivoting by Dantzig (1963) Lasdon (1970) and Cooper and Kennington (1979). The use of artificial variables to obtain a starting basic feasible solution was first published by Dantzig (1951). The revised simplex method was advised by Dantzig and Orchard-Hays (1953). The simplex method using the LU factorization of the basis was introduced by Bartels and Golub (1969). The simplex method for bounded variables was developed by Dantzig (1955) at the RAND Corporation to provide a shortcut routine for solving a problem of assigning personnel. The method was independently developed by Charnes and Lemke (1954). The simplex method for general network flow problems is a natural extension of the work on the transportation problem and the work of Koopmans (1949) relating linear programming bases for transportation problems and trees in a graph. Zadeh (1973) has exhibited the worst case exponential behaviour of the primal simplex and the primal dual network algorithm among others, using a class of modified transportation problems. Edmonds and Karp (1972) were the first to purposes a polynomial algorithm for network flow problems.

The first algorithms for shortest path problems were developed by Bell man (1958), Dijkstra (1959), Dantzig (1960) Whiting and Hiller (1960), and Floyed (1962). For the use of decomposition techniques in solving large-scale shortest path problems on special networks was proposed by Thu (1968), Shier (1973), Jarvis and Tufekci (1981). For a more detailed exposition on the relationship between shortest path algorithm & simplex method was introduced by Akgul (1986) and Zadeh (1979). The strongly feasible partition lexicographic cycling prevention rule for bounded variables linear programming problems was proposed by Cunningham (1976) and Murty (1978).

John von Neumann is credited with having first postulated the existence of a dual linear program. The dual simplex method was first developed by Lemke (1954). The primal dual algorithm was developed by Dantzig, Ford and Fulkerson (1956). The decomposition principle was introduced by Dantizig – wolfe (1960, 1961). The concept of "good" or polynomially bounded algorithms was independently proposed by Edmonds (1965) and Cobham(1965). The complexity analysis of the simplex method under various pivoting rules was introduced by Klee & Minty (1972), Jeroslow (1973), Avis & Chvatal (1978) and Goldfarb & Sit (1979).

The Soviet mathematician L.G. Khachian (1979) was the first to introduce linear programming problems in the class P of problems. Proofs of the results appear in Khachian (1981) and Gacs & Lovasz (1981). The latter paper contains the ideas of the optimal rounding scheme of Khachian's ellipsoid algorithm derives its ideas from earlier work by other soviet mathematicians including Shor (1970) and Yudin & Nemirovski (1976). Analysis of this algorithm using finite precision arithmetic appears in Khachian (1980) and Gortschel, Lovasz and Schrijver (1981). The "deep cut" variation was first presented in Shor and Gerslaovich (1979). The unbounded complexity of Khachian's algorithm with real, *non integer data was demonstrated by Traub and Wozniakowski (1982). A polynomial* varient of Kachain's algorithms of ellipsoids appears in Yaminitsky and Levin (1982). Further, ellipsoid algorithm developed by Murty (1983).

The projective polynomial time algorithm for linear programming was proposed by N. Karmarkar (1984, 1985) at At & T Bell Laboratories, U.S.A. Lawler (1985) and Pad-berg (1986) discuss the use of larger step sizes, and Gold farb and Mehrotra (1985) and Anstreicher (1986) validate techniques using approximate projection operations. Charnes, Song and Wolfe (1984) exhibit a wrost –case linear convergence rate for Karmarkar's algorithm. Iri and Imai (1986) develop a modification that is quadratically convergent. Implementation suggestions and transformations for solving general linear programs by Karmarkar's method appear in Tomlin (1985) and Shanno & Marsten (1985). Relationship of Karmarker's algorithm to projected Newton Barrier methods have been shown by Gill, Murray, Saunders, Tomblin, and Wright (1985). Ye (1987) also discusses how to obtain an optimal basis via Karmarkar's algorithm and Ye & Kojima (1987) discuss how to recover dual solutions. The affined scaling variant of Karmarkar's algorithm finds its origin in Dikin (1967, 1974) but was more recently proposed independently by Barnes (1986), and Vanderbei, Meketon & Freedman (1986). AT&T has released a commercial software based on Karmarkar's algorithm; this software implements the primal affine, scaling, the dual affine scaling and the primal dual path following variants of Karmarkar's algorithm. C.C. GonZaga (1989) developed an algorithm for solving linear programming problem in $O(n^3L)$ operations. Y. Ye (1991) developed an $O(n^3L)$ potential reduction algorithm for linear programming. D.F. Shanno (1988) developed, computing Karmarkar projections quickly. A recent contribution to an interior point method for linear programming was due to J.F. Sturm & S. Zhang (1998).

1.2.3 NONLINEAR PROGRAMMING:

Nonlinear programming emerges as an increasingly important tool in economic studies and in operation research. Nonlinear programming problems arise in various disciplines as engineering, business administration, physical sciences and in mathematics, or in any other areas where decisions must be taken in some complex situation that can be represented by a mathematical model:

$$\left.\begin{array}{l} \text{Minimize } f(x) \\ \text{Subject to, } g_i(x) \geq 0 ; \quad i = 1 \ldots\ldots\ldots m \\ \text{And} \quad x \geq 0 \end{array}\right\} \ldots\ldots(1.3.3)$$

The function f(x) or g_i (x) or both may be nonlinear functions in x.

Interest in nonlinear programming problems developed simultaneously with the growing interest in linear programming. In the absence of general algorithms for nonlinear programming problems, it lies near at hand to explore the possibilities of approximate solution by linearization. The nonlinear functions of a mathematical programming problem were replaced by piecewise linear functions, these approximations may be expressed in such a way that the whole problem is turned into linear programming.

Kuhn & Tucker (1951) published an important paper "nonlinear Programming", dealing with necessary and sufficient conditions for optimal solutions to programming problems, which laid the foundations for a great deal of later work in nonlinear programming.

If the nonlinear programming problem composed of differential objective function and equality constraints, then the optimization may be done by the use of Lagrange multipliers. A necessary condition for a function f(x) subject to the constraints g_i (x) = 0, i = 1 ...... m to have relative minimum at point x* is that the first partial derivatives of the Lagrange function defined as

$$L = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$; with respect to each of its arguments must be zero.

That is, $\frac{\delta f}{\delta x_j} + \sum \lambda i \frac{\delta g_i}{\delta x_j} = 0$; this condition is also sufficient if the quadratic from,

$$Q = \Sigma \Sigma \frac{\delta^2 L}{\delta X_I \delta X_J} dx_i dx_j,$$

Evaluated at $x=x^*$ is positive definite for all values of dx for which the constraints are satisfied. Nonlinear programming problems are in general difficult to solve. The Kuhn-Tucker conditions form the basis of many algorithms for nonlinear programming problems.

The nonlinear programming problem can be classified into various categories. Some of them are :

(i) Quadratic Programming

(ii) Convex programming

(iii) Integer programming

(iv) Stochastic programming

(v) Separable programming

(vi) Multi objective programming

(vii) Fractional programming etc.

Note that all these above categories are not mutually exclusive.

(i) Quadratic programming:

A quadratic programming problem is a nonlinear programming problem with an objective function, which is a sum of linear and quadratic forms and linear constraints. That is,

$$\text{Minimize } f(x) = c'x + x'Dx$$

$$\text{Subject to, } Ax = b$$

$$\& \ x \geq 0$$

D is an nxn symmetric matrix, A is an mxn matrix.

(ii) Convex Programming:

Minimization of a concave function subject to a set of convex constraints is known as convex programming. The standard form of a convex programming problem (CPP) may be given as:

Minimize f(x)

Subject to, $g_j(\underline{x}) \leq 0; j = 1,2,\text{-----} m$

& $\underline{x} \geq \underline{0}$

Where, f(x) is concave function and $g_1(\underline{x}), g_2(\underline{x})$, ------ $g_m(\underline{x})$ are convex functions

NOTE: Quadratic Programming Problem is a special case of Convex Programming Problem.

(iii) Integer Programming

If some or all of the decision variables of a mathematical programming problem are restricted to take only integer values, the problems is called an integer programming problem.

A linear integer programming problem can be stated as,

Minimize $f(\underline{x})$

Subject to, $g_j(\underline{x}) \geq b_i$; $i = 1 \ldots\ldots m; j= 1,2, \ldots\ldots n$

And $\underline{x} \geq 0$ & integer

If either $f(\underline{x})$ or $g_j(\underline{x})$ are nonlinear then the problem becomes a nonlinear integer programming problem.

(iv) Stochastic Programming:

A stochastic programming problem is a mathematical programming problem in which some or all of the parameter or decision variables are random variables. A stochastic liner programming problem can stated as:

$$\text{Minimize } f(x) = c'x = \sum_{j=1}^{n} c_j x_j$$

Subject to, $Ax \geq b$

$$\text{or } \sum a_{ij} x_j \geq b_i; \quad i = 1 \ldots\ldots m; \quad j = 1 \ldots\ldots n$$

Where some or all of the c_j, a_{ij} and b_i are random variables.

(v) SEPARABLE PROGRAMMING

A function f(x) of n variables is said to be separable if it can be expressed as the sum of n single variable functions $f_1(x)$, $f_2(x)$, ... $f_n(x)$, that is,

$$f(x) = \sum_{i=1}^{n} f_i(x_i)$$

A separable programming problem is one in which the objective function and the constraints are separable.

(vi) MULTIOBJECTIVE PROGRAMMING:

The simultaneous constrained optimization of more than one objective function is termed as multi objective programming. It can be stated as,

$$\text{Minimize } f_1(x), f_2(x) \ldots\ldots f_k(x)$$

$$\text{Subject to, } g_j(x) \leq 0; \quad j = 1,2 \ldots\ldots m.$$

Where $x' = (x_1 \ldots\ldots\ldots\ldots x_n)$ and k denotes the number of objective function to be minimized.

Any or all of the functions $f_i(x)$ and $g_j(x)$ may be nonlinear. The multi objective programming problem is also known as a vector minimization problem.

(vii) FRACTIONAL PROGRAMMING:

Fractional programming problem form a special class of nonlinear programming problems. A fractional programming problem can be stated as,

$$\text{Minimize } z = \frac{f(x)}{g(x)}$$

$$\text{Subject to, } h_k(x) \geq b_k; \quad k = 1......m$$

This is known as single-ratio fractional program.

A special case is the linear fractional programming problem, defined as :

$$\text{Minimize } Z = \frac{c' \; x + \alpha}{d' \; x + \beta}$$

Subject to, $\text{Ax} \geq b$

And $\quad x \geq 0$; where $d'x + \beta > 0$ assumed

1.2.4 METHOD FOR SOLUTIONS OF NLPP:

The optimization problems frequently involve nonlinear behaviour that must be taken into account, it is some times possible to reformulate these nonlinearties to fit into a linear programming format. In contrast to the case of the simplex method for linear programming, there is no efficient all purpose algorithm that can be used to solve all nonlinear programming problems. In fact, some of these problems cannot be solved in a very satisfactory manner by any method. However, considerable progress has been made for some important classes of problems, including quadratic programming, convex programming and certain special types of non-convex programming. A variety of algorithm that frequently performs well is available for these cases. Some of these algorithms incorporate highly efficient procedures for unconstrained optimization for a portion of each iteration, and some use a succession of linear or quadratic approximations to the original problem. There has been a strong emphasis in recent years on developing high quality, reliable software packages for general use in applying the best of these algorithms on mainframe computers. For example, several powerful software packages such as MINOS have been developed in the systems optimization laboratory at Stanford University.

The Karush-Kuhn –Tucker optimality conditions for nonlinear programs were developed independently by Karush (1939) in his masters thesis, and by Kuhn & Tucker (1950). An approach to generalizing the simplex method for solving the problem of minimizing a convex function subject to linear constraints is proposed by Zangwill. Gradient methods for linear and non linear constraints with convex objective function was introduced by Rosen, Zoutendijk, Wolfe, Hadley and Kunzi & Krelle. Cutting-plane methods was introduced by Kelly, Sequential unconstrained minimization was introduced by Fiacco & McCormick. Separable convex programming was introduced by Charnes and Lemke. Bari A. Arshad M. (1978) developed a Branch & Bound method for integer quadratic programming. R. Polyak & Marc Teboulle (1997) developed a Non Linear rescaling and proximal like methods in convex programming. On the complexity of approximating a KKT point of quadratic programming was developed Y.Ye. (1998). A recent contribution to an interior point method for Non-Linear programming problem was due to Wacheter and L.T. Biegler (2000).

1.3 APPLICATIONS OF MATHEMATICAL PROGRAMMING

Mathematical programming models are widely used to solve a variety of military, economic, industrial, social etc problems. In1976, survey in America to determine the use of mathematical programming in American companies shows that 74% of them use mathematical programming techniques to solve their various problems. In the following few practical situations are described which nay be formulated and solved as problems of mathematical programming.

1- RESOURCE ALLOCATION PROBLEM:

An investor wishers to invest money in several available investment choices. The returns on investment in each of these choices are known and is expected to persist in the future. The investor wishers to diversify his investment fund. Then he has a problem of allocating his funds among these choices to realized the maximum return. This is an allocation problem and can be formulated as a linear programming problem

2- PRODUCT MIX PROBLEM

A company makes n kind of products for which m basic raw materials are used. There are certain restrictions on the availability of raw materials and demand of the products in the market. The company wants to know how much of each product should be manufactured to maximize its total profit. This problem can be formulated as a mathematical programming problem to get the optimal solution.

3- BLENDING PROBLEM

Blending problem refers to situations where a number of components are mixed together to yield one or more products. There are restrictions on the available quantities of raw materials, restrictions on the quality of the products, and on the quantities of the products to be produced. The problem is, how to carry out the blending operation such that the profit is maximized.

4- DIET PROBLEM

The list of possible foods that can be included in a balanced diet is available along with their nutrient contents and the cost per unit. The problem is to find a minimum cost balanced diet

5- TRASPORTATION PROBLEM

Suppose there are m origins, which contain various amounts of commodity that has to be allocated to n destinations. Suppose commodity that has to be allocated to n destinations. Suppose there are some restrictions on the availability and demand of the commodity in the origin and destination respectively. Let the cost of shipping a unit quantity from the origin i to the destination j be known. The problem is to determine the number of units to be shipped from the origin i to the destination j so that the requirements are satisfied and the transportation cost is minimized. This problem is discussed in chapter two of this manuscript.

6- ASSIGNEMNT PROBLEM:

Let there be n workers or machine for performing n jobs, one worker/Machine can be performed only one job. Also, assume that the worker/machines vary in their respective capability and suitability to a particular job. The assignment problem is to find the best way of assigning the n workers/ machines to the n jobs.

7- TRAVELLING – SALESMAN PROBLEM:

A salesman starts from a given city and visits a group of cities. The problem is to find the shortest route for the salesman.

8- CATERER'S PROBLEM:

A caterer faced with the problem of providing napkins for dinners on each of n consecutive days. The number of napkins required on the i^{th} day is known. These requirements may be met by purchasing new napkins or by laundering napkins soiled at an earlier dinner at lesser cost. The problem is to meet the requirements for napkins at minimum cost.

9- PRODUCTION SCHEDULING PROBLEM:

A manufacturer has to produce several items. The cost of producing one item on regular time and on overtime are known. The variation of cost with time and also the capacity restrictions might make it more economical to produce in advance of the period when the items are actually needed and store them for future use at some storage cost. The problem is to determine the production schedule which minimizes the sum of production and storage costs.

10- TRIM- LOSS PROBLEM

Paper mills produce rolls of a given, standard width. Customers require rolls of various width and hence, the rolls of standard width must be cut. In general, some waste occurs at the end of the cutting process, i.e. trim loss. The manufacturer wishes to cut the rolls as ordered by his customers and to minimize the total trim loss. This application applies to similar manufacturing situations in which a standard roll, sheet etc must be cut with resulting trim loss.

11- WAREHOUSE PROBLEM

Given a warehouse with fixed capacity and initial stock of a certain product, which is subject to known seasonal price and cost variations, and given a delay between the purchasing and the receiving of the product. The problem is to find the pattern of purchasing, storage and sales, which maximizes the profit over a given period of time.

12- CRITICAL- PATH PLANNING AND SCHEDULING

A characteristic of many projects is that all work must be performed in some well-defined order, e.g. in construction work, forms must be built before concrete can be poured. This formulation concerns the scheduling of the jobs, which combine to make a project. The problem is to select the least costly schedule for desired and feasible project completion time.

1.4 SOME SPECIALISED APPLICATIONS OF MATHEMATICAL PROGRAMMING IN VARIOUS FILED

The examples given in section 1.3 are simple practical problem that arise in the work of business people, managers of industry, and other executive whose main concern is with routine operational matters. In addition to helping with the solution of problems from that sphere of activity and mathematical programming also finds numerous more abstract applications in engineering design, and scientific research. For examples from those areas of applications are given below.

1. CURUE FITTING:

Suppose that in some scientific research, say in biology or physics, a certain phenomenon $R(t)=at^2+bt+c$, is measured in the laboratory as a function of time t. The parameters a, b, and c are unknown and are to be estimated from the experimental measurements R. The problem is to find parameters a, b, and c so as to minimize the absolute values of the largest discrepancy between the measured values and corresponding theoretical values. The mathematical formulation of the problem is,

$$\text{Minimize}(\text{Maximize}\ |D_i|)$$

Where D_i is the discrepancy between measured and theoretical values. Because we are minimizing the maximum absolute deviation, such a formulation is commonly called MINIMAX problem.

2- NURSE SCHEDULING:

A hospital administrator is in charge of scheduling nurses to work in different shifts. The number of nurses required during the day is known. The problem is to schedule the nurses to meet the requirements and to do so with the minimum number of nurses. This problem can be formulated as a linear programming problem and the desired optimal solution can be obtained.

3- CONSTRUCTION OF DESIGN

In the theory of experimental design the choice of both the design and the model influence the conclusions drawn from the experiment. Thus problem of optimally choosing the design and the model may be formulated as a problem of mathematical programming. The model formulation of experimental design involves unknown parameters. These parameters can be estimated by using mathematical programming.

4- LOCATION PROBLEM

The location of a supply center to serve m customers having at fixed destinations in a city is to be selected. The commodity to be supplied from the center may be electricity, water, milk etc. the criterion for selecting the location of the supply can be formulated as a mathematical programming so as to minimize some distance function from the centre to the destinations.

5- CLUSTER ANALYSIS

Cluster analysis has been employed as an effective tool in scientific enquiry. One of its most useful roles is to generate hypothesis about category structure. The objective of cluster analysis is to design n objects to k mutually exclusive groups while minimizing some measure of dissimilarity among the items. Mathematical programming techniques are applied to these problems for minimizing these dissimilarities.

6- QUALITY CONTROL

The determination of an inspection plan for a continuous production system may be set up as a bi-criteria problem. One may be interested in minimizing the expected unit cost of inspection and replacement,. The unit cost included the cost of inspecting an item during production, the cost of replacing a defective item during inspection, and the cost of replacing defective items when returned by customers (or the next production line). The other objective may be to minimize the average outgoing percentages of defectives. Both of these objectives are nonlinear functions of decision variables. Zoints and Wallenius (1976) gave an iterative programming method for solving such problem.

7- ELECTRICITY SUPPLY SYSTEM:

Kwun & Baughman (1991) presents a model that integrates the supply planning of potential cogenerating industries with that of the electric utility suppliers. The model represents the technical alternatives for producing thermal energy in the industrial sector and electricity utility sector. The objective of the problem is to find the supply mix that minimizes the total cost of supply while meeting the required thermal electricity demands in a given system.

8- SCHEDULING OF FREIGHT TRAINS:

Freight trains over single-line track with a crossing loops needs the determination of where and when such trains should cross. The dynamic rescheduling system schedules future train movements to minimize the overall cost due to late running of trains, and energy consumption. Nonlinear programming and discreet network methods may be used to provide the train controller with continually updated crossing schedule.

9- INTEGER PROGRAMMING FOR ELECTRIC UTILITY CAPACITY PLANNING:

Mathematical programming techniques have been propose for electric utility capacity expansion planning problems as early as the work of Masse and Gibrat (1957), Linear and nonlinear programming models of deterministic type problems were used. Anderson (1972) proposed more accurate probabilistic models for determining least cost investments in electric supply. Dynamic programming approach was also used. Sherali, Staschus, Haucuz (1987) used integer programming for an electric utility capacity-planning problem.

10- OPTIMUM ALLOCATION IN STRATIFIED SAMPLE SURVEY:

The purpose of sampling theory is to develop the most economic procedures for sample selection and to obtain estimates of required precision. Sample survey problems can be formulated as optimization problems. Let n be the total sample size of stratified sample. The problem is to allocate the sample sizes $n_1, n_2 .. n_h$ to various state such that.

$$\sum_{h=1}^{L} n_h = n$$

The problem is to find the integer values of n_h by minimizing the variance for fixed cost or by minimizing the cost for fixed variance. Dynamic programming technique may be used to solve this problem. This problem is discussed in some detail in chapter five of this manuscript.

11- OPTIMUM ALLOCATION OF SURVEYS

When two or more sample surveys are conducted on the same set of units it may be stipulated that the sample units are to have different probabilities for the different surveys. If we fix the sample size n to be same for each of the surveys, a sample drawn for one survey may not in general satisfy the restrictions on the probability of selection of units for the other survey. Thus we may have to draw different samples for different surveys and increased total cost. Hence we wish to reduce the cost of surveys by drawing the maximum possible number of common units for both surveys without violating the probability restrictions. This problem can be formulated as a mathematical programming problem.

12- ENGINEERING

Mathematical programming can be applied to solve many engineering problem. Typical applications of engineering disciplines are-

(i) Design of aircraft and aerospace structures for minimum weight.

(ii) Finding the optimal trajectories of space vehicles.

(iii) Design of civil engineering structures.

(iv) Minimum weight design of structures for earthquake, wind and other types of random loading.

(v) Optimum design of electrical network etc.

13- PROJECT SELECTION

The purpose of a project selection is to assist the administrator in prioritizing and funding the available projects. Three general categories of models have been developed, (i) Checklists, (ii) economic indexes and (iii) portfolio models. Check lists and economic indexes have been developed and used largely by practicing managers and their staffs.

Portfolio models have largely been developed by operations research specialists. Portfolio models deal with the problem of determining optimum funding allocations.

14- MILITARY

A target defense problem is a very crucial problem in military especially at the time of war. It is desired to select numbers of areas and point interceptors that minimize the cost of defensive missiles under the maximum total expected damage produce by an unknown number of attacking missiles. This problem of minimizing the cost can be formulated as a mathematical programming problem. Also in planning future antiballistic missile deployments it is desirable to model the problem of determining optimal defensive deployments under some restrictions.

Caterers problem can also be viewed as a military problem where the napkins are the bombard aircraft and laundering is the repairing of these aircrafts under normal situation and at emergency period.

15- QUEUING MODEL FOR COMPUTERISED RAILWAY RESERVATION SYSTEM:

Queues arise at the railway reservation centre because of the natural and increasing demand for reservation facility. Along with improving other facilities, providing better reservation facilities is necessary for any developing nation. Since the modern amenities like computerization of the system are possible and available, the aspiration levels of the public are high. The computerization of the system involves employment of technical personnel in investment of huge funds. The management of the reservation system wishes to operate an efficient queuing system. Similarly the public wishing to get the reservation would like to know how long they have to wait in the system. So, for the efficient design and control of the reservation system appropriate queuing model is needed. Through goal programming the optional operating policies of the system and also suitable queuing model for the computerized railway reservation system was developed by K. Srinivas Rao (2001).

In the subsequent chapters some application of mathematical programming are discussed in detail

Chapter – 2

TRANSPORTATION PROBLEM

2.1 INTRODUCTION

The term 'transportation' is somewhat deceptive in that it appears to be restricted only to transportation systems. Fortunately, this is not the case. Infact many of the resource allocation problems arising in production systems can be treated as transportation problems. Typical examples of such problems are production scheduling, contract bidding, make or buy decisions and locating warehouses. A transportation model lends itself to modifications in such a way as to represent a variety of similar problems with only very little change from the standard form. Best-known extension of the standard transportation model are the transshipment and assignment models.

The basic transportation problem was originally stated by Hitchcock(1941) and Later discussed in detail by Koopmans (1947). The transportation problem is one of the sub classes of linear programming problems which the objective is to transport various amounts of single homogeneous commodity, that are initially stored at various origins to different destinations in such a way that the total transportation cost is minimum. Some additional mathematical aspects of the transportation problem and the corresponding simplex method of solution are given in "Linear Programming and Extensions" by Dantzig (1947).

2.2 FORMULATION OF THE PROBLEM

Consider m origins and n destination in a transportation system. The origin i (i = 1, 2 ----- m) has a_i units of a single commodity available to be shipped, where as the destination j (j = 1, 2 -----n) requires b_j units of the same commodity. Assume that the cost of shipping from the origin i to destination j is directly proportional to the amount of units shipped and that unit transportation cost from each origin to each destination c_{ij}, is known and constant. Then the transportation problem seeks to determine the number of units of a single commodity to be shipped from origin i to destination j, for all i and j, so that the total shipping cost can be minimized, subject to the availability and requirement constraints if we let the decision variable x_{ij}, designate the number of units of a single commodity to be shipped from origin i to destination j then the tableau shown in table 1.1 indicates the relationships between different elements of a transportation problem. This tableau will be referred to as the transportation tableau. A specific box i, j in this tableau to which an allocation of x_{ij} ($x_{ij} > 0$) units from i to j can be made will be called a cell.

Table 1.1

TRANSPORTATION TABLEAU

Destination / Origin	1	2	......	j	......	n	Available
1	C_{11} X_{11}	C_{12} X_{12}	....	C_{1j} X_{1j}	......	C_{1n} X_{1n}	a_1
2	C_{21} X_{21}	C_{22} X_{22}	......	C_{2j} X_{2j}	....	C_{2n} X_{2n}	a_2
. .			....		......		
i	C_{i1} X_{i1}	C_{i2} X_{i2}	......	C_{ij} X_{ij}	....	C_{in} X_{in}	a_i
. .			....		......		
m	C_{m1} X_{m1}	C_{m2} X_{m2}	......	C_{mj} X_{mj}	....	C_{mn} X_{mn}	a_m
Required	b_1	b_2	....	b_j	......	b_n	

On the basis of this description, the general model of the transportation problem can be formulated as follows:

$$\text{Minimize } z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to, } \left.\begin{aligned} &\sum_{j=i}^{n} x_{ij} = a_{i,} \qquad i = 1,\ldots.,m \\ &\sum_{i=1}^{m} x_{ij} = b_{j,} \qquad j = 1,\ldots n \end{aligned}\right\} m+n \text{ Constraints}$$

Where $x_{ij} \geq 0$ and a_i & b_j are positive.

In order for the problem to have a feasible solution it is obvious that the total availability must be equal to the total requirement thus we have the following additional constraints:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

A transportation problem for which the later constraint is satisfied is called a balanced transportation problem otherwise the problem is called an unbalanced transpiration problem. In general, the model shown here is called the standard transportation problem. The term standard is used here to differentiate the classical problem initiated by Koopman & Hitchcock forms its extensions.

2.3 EXTENSIONS OF THE TRANSPORTATION PROBLEMS:

Since the appearance of the transportation problem in 1941 and its efficient solution by the simplex method in 1947, several Researchers have defined new problem and introduce models and algorithms that are very similar in structure to those of the standard transportation problem. Two extensions are transshipment and assignment problem of standard transportation problem. Further the following extensions of the transportation problem are:

(A) THE TRANSSHIPMENT PROBLEM:

The Hitchcock – Koopman standard transportation problems was extended and solved by Orden (1956) to include the possibility of transshipment.

The basic ideal underlying the solution method is, first to convert the transshipment problem to a transportation problem. A transshipment tableau differs from a transportation tableau in that in the former each origin and each destination of the problem is permitted to act as an origin and destination at the same time.

(B) THE ASSIGNMENT PROBLEM:

The assignment problem is one of the special types of transportation problem for which more efficient (less time-consuming) solution methods have been device.

Consider n jobs and n candidates for the jobs. Assume that each job can be performed only by one candidate and that the effectiveness of the i^{th} candidate (i = 1, 2, ... n) in performing the j^{th} job ($j = 1, 2 \ldots\ldots., n$) is measured by a measure of effectiveness represented by a real number c_{ij}. On this basis, the assignment problem deals with the assignment of n candidates to n jobs in such a way as to optimize the sum of the measure of effectiveness of individual candidates.

Let us define x_{ij} as the i^{th} candidate assigned to j^{th} job then the model of the assignment problem can be stated as follows:

$$\text{Optimize } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Where } x_{ij} = \left\{\begin{array}{l} 0, \text{ If job i is not assigned to candidate j} \\ 1, \text{ If job i is assigned to candidate j} \end{array}\right\}$$

Subject to restrictions:

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (i = 1\ldots\ldots,n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad (i = 1,\ldots\ldots,n)$$

$$x_{ij} = 0 \text{ or } 1$$

Obviously, the assignment problem is a special case of the standard transportation problem with the following properties:

1- The number of origins is equal to the number of destinations, that is, $m = n$

2- Each origin is represented by one candidate that is, $a_i = 1$ where $i = 1$ ------, n

3- Each destination is represented by one job, that is, $b_j = 1$ where $j = 1$, --------, n.

The solution of the assignment problem by the transportation technique would require $m + n - 1$ or $2n - 1$ occupied cells at each iteration.

1- GENERALIZED TRANSPORTATION PROBLEM:

The Generalized transportation model problem is defined by the following model

$$\text{Minimize} \quad z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\text{Subject to, } \sum_{j=1}^{n} d_{ij}x_{ij} \pm s_i = a_i, i = 1.......,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = 1.......,n$$

$$x_{ij} \geq 0 \text{ And } s_i \geq 0 \text{ (Slack or surplus variables)}$$

The coefficients of the decision variables (x_{ij}'s) in the availability constraints are the arbitrary number (d_{ij}) rather than unity and the availability constraints are not strict equality constraints.

This type of problems arising in production system, Assume that there are a given number of different machines, i = 1, 2, ..., m, each with a specific number of hours a_i available for production during a given production period. There are also j = 1 , 2, ...,n, number of distinct products, each to be produced in specific amount b_j during the same period. Each unit of product j consumes d_{ij} hours of the i^{th} machine with a corresponding production cost c_{ij}, the problem is to determine how much of each product i should be produced on each machine j (i.e. x_{ij}) such that all production requirements can be satisfied for the given production period at a minimum total production cost.

2- CAPACITATED TRANSPORTATION PROBLEM:

The Capacitated transportation model differs from the standard transportation model in that it considers capacity constraints in the transportation process in addition to the availability and requirement constraints. A typical area where the capacitated transportation model can be applied is in the distribution of goods where there may be finite shipping capacities over some or all of the routes for instance, the number of trucks, trailers, or aircraft available. A typical example can be found in the automobile industry.

3- TRANSPORTATION PROBLEM WITH MIXED CONSTRAINTS:

There are many situations in which this type of problem may arise. For example, introduction planning for a given production period during which an upward trend of demand is expected for certain products, it would be economically more advantageous to plan the raw material supply in atleast the $\geq$ mode. Similarly, for products with downward trend, to plan material supply in at most the $\leq$ mode would be more prudent. On the other hand, for products where demand is set by contractual agreements, the equality type of demand supply relationship would be preferred.

4- TRANSPORATION PROBLEM WITH QUANTITY DISCOUNT:

In this problem, per unit transportation costs from an origin i to a destination j are subject to quantity discounts, such that there exist finite intervals for quantities where price breaks are offered to customers. The availability and requirement constraints of the standard transported problem remain the same for this problem.

5- FIXED CHARGE TRANSPORTATION PROBLEM:

In this problem a fixed charge f_{ij} is associated with each route that can be opened to ship goods from an origin i to a destination j, in addition to the variable transportation cost, which is proportional to the number of goods shipped the availability and requirement constraints of the standard transportation problem remain the same for fixed charge problem.

6- SINGLE SOURCE TRANSPORTATION PROBLEM:

A single source T.P. is STP with the additional constraints that the entire demand at each demand location be supplied from a single supply location. The single source T.P. arises in the assignment of jobs to computers, (Blachandran 1976), facility location problem (Ross & Soland 1977), and machine loading problems (Christofides & Eilon 1971)

7- THE TRI DIAGNOL TRANSPORTATION PROBLEM:

The tri diagonal T.P. is a special case of the STP in that the transformation tableau contains allocations in the cells along the main diagonal, a band above it, and a band below it, while the other cells of the tableau are infinitely costly and hence prohibitive for any allocation. For a balanced STP tableau with origins and destinations both equal to n, this occurs for any cell (i, j) of the transportation tableau for which $|i-j| \leq 1$.

8- TRANSPORTATION LOCATION PROBLEM:

The TLP involves the establishment of the locations and capacities of plants or distribution centers that will produce and/or distribute products to retail or wholesale outlets with known requirements for a given product.

9- THE TIME MINIMIZING (BOTTLENECK) T.P.:

This problem arises in situations where the items (such as perishable goods, military equipment, and aircraft) are sifted from origins to destinations in such a way as to minimize the maximum transportation time associated between an origin and a destination. A typical time minimizing T.P. may be taken form the rescue missions planned to assist disaster areas.

10- T. P. WITH COST/COMPLETION DATE TRADE OFFS:

In this problem the STP is reconsidered to include the objective of minimizing the date at which all shipments are completed, in addition the usual objective of minimizing total shipping cost. It is a modification of the time – minimizing T.P.

11- BI CRITERIA T.P:

As in the time minimizing and cost/completion date trade off T.P., the bi criteria T.Ps. also consider an additional objective to the cost objective of the S.T.P. and both objectives in linear functions of the objective criteria.

12- MULTIPLE GOAL TRANSPORTATION PROBLEM:

In Multiple Goal Transportation problems, more than one objective can be considered. However, these objectives are formulated as goal constraints, while the objective function of the model is formulated in such a way as to minimize the deviations from the set goals.

13- MULTI-INDEX T.P.

The STP is a two dimensional problem that considers the minimum cost transportation of one and the same kind of goods from certain origins to certain destinations under the availability and requirement constraints. The multi-index T.P. adds a third dimension to the S.T.P and makes it possible to consider more than one kind of goods for shipment.

14- NON – LINEAR T.P.:

This problem differs form the STP in that in the objective function, alongwith the total linear transportation cost of the product, it also considers the total non-linear cost of production of the product produced in each of the origin. This additional non-linear cost function is continuous and non-decreasing for all the origins. The objective is to find the amount of the product to be supplied from each origin to each destination, such that the total transportation and production costs are minimized.

15- LARGE-SCALE T.P.

The Transportation Problems arise in very large sizes, containing many hundreds of constraints and variables. For this reason the solution of such problems on a computer requires excessive computational time.

2.4 METHODS OF SOLUTIONS:

The stepping stone method was presented by Cooper and Charnes (1954) as a simplification to the original modified simplex solution to the transportation problem suggested by Dantzig (1947). Vogel (Reinfeld & Vogel 1958) has devised a method that assures a basic feasiblesolution close to the optimum. The assignment problem will be solved by a method thatoriginated with Kuhn (1956) and Flood (1956). The justification of the steps leading to the solution is based on theorems proved by Hungarian-mathematicians Koenig (1950) and Egervary (1953), hence the method is called Hungarian. An algorithm based on the dual variable method of the generalized T.P. can be found in Dantzig (1963), Hadley (1963) and Taha (1968). Solutions based on the stepping stone method have been suggested by Balas and Ivanescu (1964), Lourie (1964) and Eisman (1964). A computer program for the solution of GTP can be found in Eisman and Lourie (1959).

Several methods have been suggested for the solution of the capacitated T.P. that are more efficient can be found in Dantzig (1963), Hadley 91963), Wagner (1975), Simonnard (1966), and Spivey and Thrall (1970). Klingman and Russel (1977 have presented a method that transforms the transportation problem with mixed constraints to an equivalent standard T.P. for which different solution methods are available. Balachandran and Perry (1976) have presented algorithm for the all unit quantity discount T.P. and the incremental quantity discounts T.P. respectively. Gray (1971) has presented an algorithm for the exact solution of the fixed charge transportation problem. Barr, Glover and Kling man (1981) have proposed a branch and bound algorithm for solving fixed charge T.P. Applied studies in the fixed charge transportation model have been reported by Jarvis, Unger, Radin and Moore (1978); Stroup (1967); Spielberg (1970); and Kennington (1976). Nagelhout and

Thompson (1980) have presented two heuristic methods and an algorithm for solving the single T.P. Lev(1977) developed and algorithm that yields the optimal solution to such tri diagonal problems in n steps for an n origin, n-destination problem. The transportation location problem was formulated by Cooper (1963), who also provided exact and approximate methods for the solution of the problem in (1972).

Garfinkal and Rao (1971) developed two algorithms for the solution of the time-minimizing T.P. An extensive survey of the solution of this problem was providing by Szware (1971). Garfinkla and Rao (1971) have reported that their algorithms have been programmed in FORTRON.

A procedure to generate all feasible trade off for small-size problems was provided by Glickman & Berger (1977). Aneja and Nair (1979) provided a simpler method for generating optimal solutions for bi criteria T.Ps. from which the best transportation plan can be selected by the decision maker. Lee and Moore (1973) used goal-programming techniques to solve multiple goal transportation problems. Further publications in this area have been contributed by Hill (1973); Kwake and Schniederjans (1979); and Moore, Taylor, and Lee (1978). Haley (1962) developed an algorithm for the solution of the multi index T.P.. Further publications in this area have been contributed by Moravek and Vlach (1967), Haley (1967) and Smith (1968). A graphical solution has been developed by Vidale(1956) for the non-linear transportation problem. Williams (1962) presented a method for the decomposition of large-scale transportation problems.

One of the first extensions of the transportation model to be used in production scheduling was proposed by Bowman (1956) and Fetter (1967). Comments and extensions on the use of the transportation model in production scheduling were contributed by Elmaghraby (1957), Bishop 1957, Jhonson (1957), and Kunreuther (1971). Basically these papers discuss the properties of the transportation model as applied to production scheduling and proposed certain rules and procedures to simplify and reduce computation for the solution. Rand (1974) proposed a manual algorithm for the transportation model for simple and multi product scheduling problems without backlogging. Klein (1983) has extended the transportation model to include both inventory and back order constraints. Evans (1985) has related Klein's worked to Wagner's (1959) general method and has shown that a variety of production planning problems can be formulated as transportation problems.

2.5 A NUMERICAL EXAMPLE

The perfect mixer company produces mixers to be used for mixing drinks the company maintains warehouses throughout the country that supply the major distributors of the mixers on a weekly basis. For the next week, the warehouses in San Antonio, Cleve lard and Los Angels, which will be identified as warehouses, 1, 2, and 3 respectively, will supply the distributors in Chicago, New Orleans, New York and El Paso designated have as A, B, C and D respectively. The three warehouses have a stock of certain model of mixers in 15, 15 and 25 units, while distributors require 20, 15 , 10 and 10 units of this model respectively. The unit shipping costs between warehouses and distributors are shown in Table 1.2.

How many mixers should be shipped from each warehouse to each distributor so that the requirements of the distributors can be met at minimum shipping cost?

The transportation tableau of this problem is shown in table 1.3, obviously, it is a balanced transportation problem.

Table 1.2
UNIT SHIPPING COST – PERFECT MIXER

Distributor / Warehouse	(A) Chicago	(B) New Orleans	(C) New York	(D) El Paso
(1) San Antonio	$ 5	3	6	1
(2) Cleveland	2	7	4	8
(3) Los Angles	4	6	9	2

Table 1.3
TRANSPORTATION TABLEAU OF THE EXAMPLE PROBLEM

Distributor / Warehouse	A	B	C	D	Available
1	5 X_{1A}	3 X_{1B}	6 X_{1C}	1 X_{1D}	15
2	2 X_{2A}	7 X_{2B}	4 X_{2C}	8 X_{2D}	15
3	4 X_{3A}	6 X_{3B}	9 X_{3C}	2 X_{3D}	15
Required	20	15	10	10	55

The model of the transportation problem can be formulated as in the following

Minimize $z = 5x_{1A} + 3x_{1B} + 6x_{1C} + \dots\dots\dots + 2x_{3D}$

Subject to:

$$\text{Availabilities}\left\{\begin{array}{lllll} x_{1A}+x_{1B}+x_{1C}+x_{1D} & & & = 15 \\ & x_{2A}+x_{2B}+x_{2C}+x_{2D} & & = 15 \\ & & x_{3A}+x_{3B}+x_{3C}+x_{3D} & = 25 \end{array}\right.$$

$$\text{Requirements}\left\{\begin{array}{llllllllllllll} x_{1A} & & & & x_{2A} & & & & x_{3A} & & & & = 20 \\ & x_{1B} & & & & x_{2B} & & & & x_{3B} & & & = 15 \\ & & x_{1C} & & & & x_{2C} & & & & x_{3C} & & = 10 \\ & & & x_{1D} & & & & x_{2D} & & & & x_{3D} & = 10 \end{array}\right.$$

Where $x_{ij} \geq 0$ (i = 1, 2, 3; j= 1, 2, 3, 4)

REVISED SIMPLEX METHOD FOR SOLUTION OF THE BALANCED TRANSPORTATION PROBLEM

For the solution of the balanced transportation problem the same basic stages are required as in the simplex solution of the linear programming problem.

2- Find an initial basic feasible solution
3- Improve the initial solution by further iterations until the optimum solution is reached.

Table 1.4
INITIAL SOLUTION TABLEAU
(North West Corner Method) (Z = $295)

Distributor / Warehouse	A	B	C	D	Available
1	$\boxed{5}$ 15	$\boxed{3}$	$\boxed{6}$	$\boxed{1}$	15
2	$\boxed{2}$ 5	$\boxed{7}$ 10	$\boxed{4}$	$\boxed{8}$	15
3	$\boxed{4}$	$\boxed{6}$ 5	$\boxed{9}$ 10	$\boxed{2}$ 10	25
Required	20	15	10	10	**55**

ITERTIONS FOR TRANSPORTATION PROBLEM BY APPLICATION OF THE STEPPING STONE METHOD

Iteration in the transportation problem consists of the following four steps:

1- Select the entering variable (optimality test and optimality principle)

2- Select the leaving basic variable (feasibility principle).

3- Restore the new transportation tableau of the new basic solution.

4- Repeat the iterations (i.e., steps 1 through 3) until the optimal solution is found.)

Table 1.5
FIRST ITERATION TABLEAU (Stepping Stone Method)
Z = $ 225

Destination / Origin	A	B	C	D	Available
1	5 (cost 5)	10 (cost 3)	(cost 6)	(cost 1)	15
2	15 (cost 2)	(cost 7)	(cost 4)	(cost 8)	15
3	(cost 4)	5 (cost 6)	10 (cost 9)	10 (cost 2)	25
Required	20	15	10	10	**55**

Table 1.6
SECOND ITERATION TABLEAU (Stepping Stone Method)
Z = $ 205

Destination / Origin	A	B	C	D	Available
1	(cost 5)	15 (cost 3)	(cost 6)	(cost 1)	15
2	15 (cost 2)	(cost 7)	(cost 4)	(cost 8)	15
3	5 (cost 4)	ε (cost 6)	10 (cost 9)	10 (cost 2)	25
Required	20	15	10	10	**55**

Table 1.7
THIRD ITERATION TABLEAU (Stepping Stone Method)
Z = $ 175

Destination / Origin	A	B	C	D	Available
1	(cost 5)	15 (cost 3)	(cost 6)	(cost 1)	15
2	5 (cost 2)	(cost 7)	10 (cost 4)	(cost 8)	15
3	15 (cost 4)	ε (cost 6)	(cost 9)	10 (cost 2)	25 + ε
Required	20	15 + ε	10	10	**55 + ε**

Table 1.8
TRANSPORTATION TABLEAU FOR OPTIMAL SOLUTION
Z = $ 175

Destination / Origin	A	B	C	D	Available
1	[5]	15 [3]	[6]	[1]	15
2	5 [2]	[7]	10 [4]	[8]	15
3	15 [4]	[6]	[9]	10 [2]	25
Required	20	15	10	10	**55**

The information contained in the optimal transportation tableau in table 1.9 provides the basis for the optimal transportation plan (shown in fig. 1.1) to be used by the distribution section of the perfect – Mixer Company.

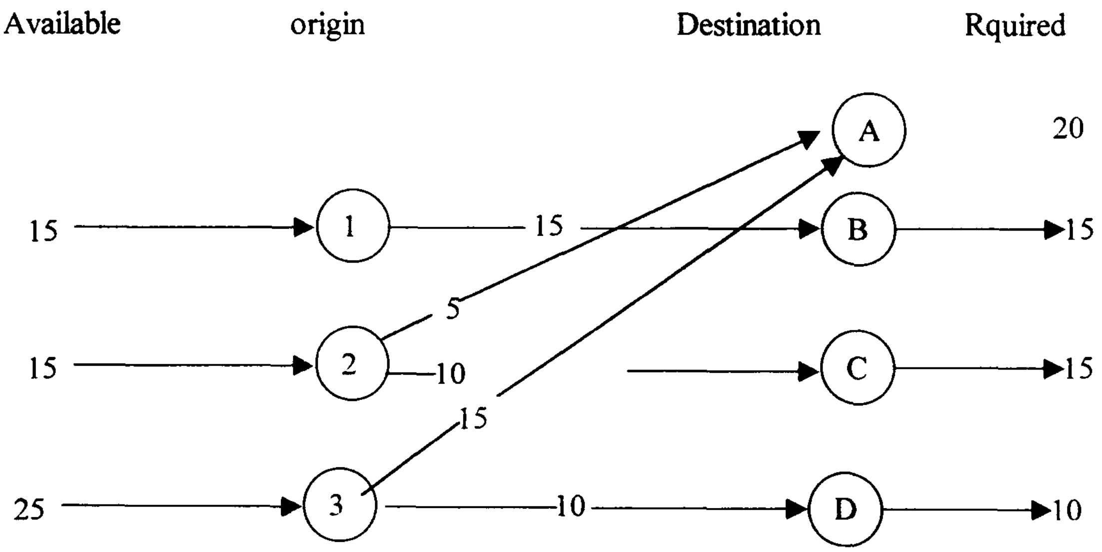

Figure 1.1
Optimal transportation plan for the example problem

2.6 RELATION OF TRANSPORTATION PROBLEMS WITH NETWORK PROBLEMS

Transportation and assignment problem are a pair of important network structured linear programming problems, that arise in several contexts and that have deservedly received a great deal of attention in literature. Although these problems are special case of network flow problems, that any finite valued, capacitated or un capacitated minimum cost network problem can be transformed into an equivalent transportation problem.

Infact, although, the assignment problem is itself a special case of transportation problem. Any transportation problem and hence any minimum cost network flow problem can be equivalently transformed into an assignment problem.

Barr, Glover & Klingman (1981) have propose a branch & bound algorithm for solving fix charged transportation problems, that have fixed charges only a portion of the existing routes (branches) of the transportation network.

2.7 DISCUSSION:

It is difficult to state which of all the available procedure for solving the transportation problems is most efficient in terms of computational time, so we can try to developed more efficient and economical procedure for the solution of the transportation problems and its variations.

Chapter-3

GAME THEORY

3.1 INTRODUCTION

Many practical problems require decision making in a competitive situation where there are two or more opposing parties with conflicting interests and where the action of one depends upon the one taken by the opponent. For example, candidates for an election, advertising and marketing campaigns by competing business firms, countries involved in military battles, etc. have their conflicting interests. In a competitive the courses of action (alternatives) for each competitor may be either finite or infinite. A competitive situation will be called a 'Game' if it has the following properties:

(i) There are a finite number of competitors (participants) called players.

(ii) Each player has a finite number of strategies (alternatives) available to him.

(iii) A play of the game takes place when each player employs his strategy.

(iv) Every game results in an out come. e.g., loss or gain usually called pay off, to some player.

3.1.1 BRIEF HISTORY OF GAME THEORY

Game theory started with two papers by Von Neumann (1928, 1937). It really sprang to life, however, with the publications of Von Neumann and Morgenstren's book theory of games and Economic Behaviour in 1944, and second edition in 1947. The reason was mainly the Second World War, because during this there had been considerable activity in modeling decision situations, which involved one or more decisions-makers. Hence the rise of O.R. most of the military problems that can be modeled as games are of the two-player zero-sum type, and these are the very ones for which game theory can suggest a specific 'solution'. Thus, at the end of the war, people were thinking of how to model decision situations and there was a view that game theory had a successful if secret, track record in the military area.

The four volumes: contributions to the theory of Games (Kuhn and Tucker,1950, 1953; Drescher, Tucker and Wolfe1957; and Tucker & Luce 1959), give a good idea of the problems that were being examined at that time. Luce & Raiffa wrote the other classic book "Games and Decisions in 1957. Lucas (1967) was proposed a ten person game. Dilemma is a popular candidate for a game, and the number of published papers concerning experiments with it is over two hundred gaming was introduced by Dilemma the idea of metagames introduced by Howard (1971) Maynard – Smith (1974) described how a game theoretic framework is useful in describing the evolution of genes, which affect breeding patterns. K.T. Lee & K.L. Teo (1993) developed a game with distorted information. Arthur T. Benjamin and A.J. Glodman (1994) developed localization of optimal strategies in certain games. A recent contribution to game theory was due to X. Deng, T.Ibaraki, H. Negamochi & W. Zang (2000).

Thus, games & game theory can look forward to an exciting future, not as a way of solving all conflict problems, but as the most useful collection of techniques for analyzing these problems.

3.2 TWO-PERSON ZERO-SUM GAMES:

When there are two competitors playing a game it is called 'two – person game'. In case the number of competitors exceeds two say n, then the game is termed as a n person game'.

Games having the 'Zero-sum' character that the algebric sum of gains and losses of all the players is zero are called zero-sum games with two players are called two-person zero-sum games. In this case the loss (gain) of one player is exactly equal to the gain (loss) of the other. If the sum of gains or losses is not equal to zero, then the game is of non-zero sum character or simply a non-zero sum game.

3.2.1 SOME BASIC TERMS

(i) **Player:** The competitors in the game are known as players. A player may be individual or group of individuals, or an organization.

(ii) **Strategy:** The strategy of a player is the pre-determined rule by which a player decides his course of action from his own list of course of action during game. Strategy may be of two types.

(a) Pure Strategy: If the players select the same strategy each time, then it is referred to as pure strategy. In this case each player knows exactly what the other players is going to do. The objective of the players is to maximize gains or minimize losses.

(b) Mixed Strategy: A mixed strategy is a decision, in advance of all plays, to choose a course of action for each play in accordance with some particular probability distribution. The objective of the players is to maximize expected gains or to minimize losses.

Note: pure strategy is special case of mixed strategy.

(iii) Optimum Strategy: A course of action or play, which puts the players in the most preferred position, irrespective of the strategy of this players is called an optimum strategy.

(iv) Value of the game: It is expected pay-off of play when all the players of the game follow their optimum strategy. It is generally denoted by v and is unique if $v = 0$, it is called a fair game.

(v) Pay off matrix: When the players select their particular strategies, the pays off (gains or losses can be represented in the form of a matrix called the pay off matrix. Since the game is zero-sum, therefore gain of one player is equal to the loss of other and vice – versa.

3.2.2 THE MAXIMIN – MINIMAX PRINCIPLE:

In the game theory the best optimal strategies for each player are determined on the basis of Maximin – Minimax criterion of optimality.

For player A, minimum value in each row represent the least gain (payoff) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that maximizes his minimum gains this choice of player A is called the maximin - principle, and the corresponding gain is called the maximum value of the game.

For player B, on the other hand, likes to minimize his loses. The maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the matrix by column maxima. He will then select the strategy that minimize his maximum losses, this choice of player B is called the minimax - principle, and the corresponding losses is the minimax value of the game.

Note: The maximin value equal the minimax value, then the game is said to have a ***saddle (equilibrium) point*** and the corresponding strategies are called ***optimum strategies***. The amount of pay off at an equilibrium point is known as the ***value of the game***.

Theorem 3.2.1 : Let (a_{ij}) be the m x n pay-off matrix for a two person zero sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ the minimax value of the game then $\bar{v} \geq \underline{v}$ that is,

$$\min_{1 \leq j \leq n}\left[\max_{1 \leq i \leq m}\{a_{ij}\}\right] \geq \max_{1 \leq i \leq m}\left[\min_{1 \leq j \leq n}\{a_{ij}\}\right]$$

RULES FOR DETERMINING A SADDLE POINT:

1- Select the minimum element of each row of the pay off matrix and mark them [*]

2- Select the greatest element of each column of the pay off matrix and mark them [+]

3- If the appears an element in the pay of matrix marked [*] and [+] both, the position of that element is a saddle point of the pay off matrix.

3.2.3 GAME WITHOUT SADDLE POINTS – MIXED STRATEGIES:

As determining the minimum of column maxima and the maximum of row minima are two different operations, there is no reason to expect that they should always lead to unique payoff position i.e. the saddle point.

In all such cases to solve games, both the players must determine an optimal mixture of strategies to find a saddle (equilibrium) point. The optimal strategy mixture for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called ***mixed strategies*** because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents which least player A can expect to win and the least which player B can lose. The expected pay off to a player in a game with arbitrary payoff matrix (a_{ij}) of order m x n is defined as:

$$E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j = p^T A q$$

Where p and q denote the mixed strategies for players A and B respectively.

Maximin-Minimax Criterion: Consider an m x n game (a_{ij}) without any saddle point. i.e. strategies are mixed. Let p_1 p_2, ...p_m be the probabilities with which player A will play his moves A_1, A_2 ....A_m respectively; and let q_1, q_2 ......q_n be the probabilities with which player B will play his moves B_1, B_2......B_n respectively. Obviously, $p_i \geq 0$ (i = 1, 2,....m), $q_j \geq 0$ (j = 1, 2, ... n), and $p_1 + p_2 + ...+ p_m = 1$; $q_1 + q_2 + ... + q_n = 1$.

The expected payoff function for player A is

$$E(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j$$

Making use of maxmin – minimax criterion, we have

For player A,

$$\underline{v} = \max_{p} \min_{q} . E(p,q) = \max_{p}\left[\left\{\min_{j} . \sum_{i=1}^{m} p_i a_{ij}\right\}\right]$$

$$= \max_{p} . \left[\min_{j} . \left\{\sum_{i=1}^{m} p_i a_{i1} \quad \sum_{i=1}^{m} p_i a_{i2} \quad \cdots\cdots \quad \sum_{i=1}^{m} p_i a_{in}\right\}\right]$$

Here, $\min_{j} . \left\{\sum_{i=1}^{m} p_i a_{ij}\right\}$ denotes the expected gain to player A when player B uses his j^{th} pure strategy.

For Player B,

$$\bar{v} = \min_{q} . \left[\max_{i} . \left\{\sum_{j=1}^{n} q_j a_{1j} \quad \sum_{j=1}^{n} q_j a_{2j} \cdots\cdots \sum_{j=1}^{n} q_j a_{mj}\right\}\right]$$

Here, $\max\left\{\sum_{j=1}^{n} q_j a_{ij}\right\}$ denotes the expected loss to player B when player A uses his i^{th} strategy.

The relationship $\underline{v} \leq \bar{v}$ holds goods in general and when p_i and q_j correspond to the optimal strategies the relation holds in 'equality' sense and the expected value for both the players becomes equal to the optimum expected value of the game.

Note: A pair of strategies (p, q) for which $\underline{v} = \bar{v} = v$ is called a saddle point of E (p, q)

Theorem- 3.2.2: For any 2x2 two - person zero-sum game without any saddle point having the payoff matrix for player A,

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

the *optimum mixed strategies*

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \quad \text{and} \quad S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

are determined by

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \qquad \frac{q_1}{q_2} = \frac{a_{22} - a_{a2}}{a_{11} - a_{21}}$$

where $p_1 + p_2 = 1$ and $q_1 + q_2 = 1$. the value v of the game to A is given by

$$v = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Note: The various methods can be used for determining the mixed strategies and value of the game. The basic approach was first to find out a saddle point for a given game and thus obtain a pure strategy. In problems where a saddle point does not exist, the simple algebric approach will assist as to determine optimal mixed strategies for games of 2x2. In problems where the pay – off matrix is of mx2 or 2xn, the graphical method will help us to find a solution. When a game matrix can not be reduced in size (2x2) by graphical method or by using the principle of dominance, then for nxn games, we can determined the solution by using the arithmetic method. For problems of higher order pay-off matrices (i.e., 3×3 or higher) with no saddle points or no dominance, linear programming approach offers the best solution.

3.3.0 SOLUTION OF A GAME BY LINEAR PROGRAMMING APPROACH:

We shall now give a general approach to solve a game by linear programming technique. To illustrate the connection between a game problem an a linear programming problem, let us consider an m x n payoff matrix (a_{ij}) for player A. Let

$$S_m = \begin{bmatrix} A_1 \ldots\ldots & A_m \\ p_1 \ldots\ldots & p_m \end{bmatrix} \quad \text{and} \quad S_n = \begin{bmatrix} B_1 \ldots\ldots & B_n \\ q_1 \ldots\ldots & q_n \end{bmatrix} \text{where} \sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1.$$

be the strategies for the two players respectively.

Then, the expected gains g_j (j = 1,......,n) of player A against B's pure strategies will be

$$\begin{aligned} g_1 &= a_{11} p_1 + a_{21} p_2 + \ldots\ldots\ldots + a_{m1} p_m \\ g_2 &= a_{12} p_1 + a_{22} p_2 + \ldots\ldots\ldots + a_{m2} p_m \\ &\vdots \\ g_n &= a_{1n} p_1 + a_{2n} p_2 + \ldots\ldots\ldots a_{mn} p_m \end{aligned}$$

and the expected losses i, (i = 1.......m) of player B against A's pure strategies will be

$$\begin{aligned} l_1 &= a_{11} q_1 + a_{12} q_2 + \ldots\ldots\ldots + a_{1n} + q_n \\ l_2 &= a_{21} q_1 + a_{22} q_2 + \ldots\ldots\ldots + a_{2n} q_n \\ &\vdots \\ l_m &= a_{m1} q_1 + a_{m2} q_2 + \ldots\ldots + a_{mn} q_n \end{aligned}$$

The objective of player a is select p_i (i = 1, 2, ...,m) such that he can maximize his minimum expected gains and the player B desires to select q_j (j = 1, 2, ...n) that will minimize his expected losses.

Thus if we let $u = \min_j . \sum_{i=1}^{m} a_{ij} p_i (j = 1,2,\ldots\ldots.m)$ and $v = \max_i \sum_{j=1}^{n} a_{ij} q_i (i = 1,2,\ldots.m)$

The problem of two players could be written as;

Player A

Maximize u = Minimize $\frac{1}{u} = \sum_{i=1}^{m} \frac{p_i}{u}$

Subject to the constraints,

$$\sum_{i=1}^{m} a_{ij} p_i \geq u \quad \text{and} \quad \sum p_i = 1, \;\; p_i \geq 0 \;\; (i = 1,2,....m)$$

Player B

Minimize v =maximize $\frac{1}{v} = \sum_{j=1}^{n} \frac{q_j}{v}$

Subject to the constraints,

$$\sum_{j=1}^{m} a_{ij} q_j \leq v \text{ and } \sum q_j = 1, q_j \geq 0 \;\; (j = 1,2....,n)$$

Assuming that u>0 and v>0, we introduce new variables defined by

$p_i' = p_i / u$ and $q_j' = q_j / v$ $\qquad\qquad (i = 1,2,...m; j = 1,2...n)$

then the pair of linear programming problems can be re-written as:

Player A

Minimize $p_0 = p_1' + p_2' + ... + p_m'$

Subject to the constraints,

$$a_{ij} p_1' + a_{2j} p_2' + ... + a_{mj} p_m' \geq 1$$

$$p_i' \geq 0 \quad (i = 1,2....,m; j = 1,2.....,n)$$

Player B

Maximize $q_0 = q_1' + q_2' + ... + q_n'$

Subject to the constraints,

$$a_{ij} q_1' + a_{i2} q_2' + ... + a_n q_n' \leq 1$$

$$q'_j \geq 0 \quad (i = 1,,....m; j = 1,2....,n)$$

Remarks

1: It is easy to note that the L.P.Ps. of the two players represent a primal – dual pair. Therefore by fundamental theorem of duality one can read off the optimal solution of one player, just from the optimum simplex table of the opponent. That is, we need to solve just one player's L.P.P. by simplex method.

2: Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negative value v of the game, the data to the problem, i.e. a_{ij} in the payoff table should all be non-negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero, the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

3.3.1 AN EXAMPLE:

Two firms are competing for business under the condition so that one firm's gain is another firm's loss. Firm A's pay off matrix is given below:

$$\begin{array}{cc} & \text{Firm B} \\ \text{Firm A} & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

Suggest optimum strategies for the two firms by linear programming technique.

Solution: Since some of the entries in the pay off matrix are negative, we add a suitable constant to such of the entries to ensure them all positive. Thus, adding a constant c=4 to each element, we get the following revised payoff matrix.

$$\begin{array}{cc} & \text{Firm B} \\ \text{Firm A} & \begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix} \end{array}$$

Let the strategies of the two firms be

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ P_1 & P_2 & P_3 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

Where, $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

The linear programming formulation for the two firms problem are:

For Firm A,

Minimize v $\equiv$ Maximize $\frac{1}{v} = x_1 + x_2 + x_3$

Subject to the constraints,

$$5x_1 + 7x_2 + 10x_3 \geq 1,$$

$$3x_1 + 9x_2 + 6x_3 \geq 1.$$

$$7x_1 + x_2 + 2x_3 \geq 1$$

and $\quad x_j \geq 0 \quad (j = 1,2,3)$

For Firm B,

Minimize v $\equiv$ Maximize $\frac{1}{v} = y_1 + y_2 + y_3$

Subject to the constraints,

$$5y_1 + 3x_2 + 7y_3 \leq 1,$$

$$7y_1 + 9y_2 + y_3 \leq 1.$$

$$10y_1 + 6y_2 + 23_3 \leq 1$$

and $\quad y_j \geq 0 \quad (j = 1,2,3)$

Where, $x_j = p_j / u (j = 1,2,3)$ and $y_j / v (j = 1,2,3)$; u = minimum expected gain to A and v= minimum expected loss to B.

Let us now solve the problem for firm B. By introducing slack variables $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 \geq 0$; the iterative simplex table are

Initial iteration. (Introduce y_3 and drop y_4.)

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	1	5	3	7	1	0	0
0	y_5	1	7	9	1	0	1	0
0	y_6	1	10	6	2	0	0	1
	$\frac{1}{v}$	0	-1	-1	-1	0	0	0

First Iteration. (Introduce y_2 and drop y_5.)

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	1/7	5/7	3/7	1	1/7	0	0
0	y_5	6/7	44/7	60/7	0	-1/7	1	0
0	y_6	5/8	60/8	36/7	0	-2/7	0	1
	$\frac{1}{v}$	1/7	-2/7	-4/7	0	1/7	0	0

Final iteration. (Optimum Solution.)

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
0	y_3	1/10	2/5	0	1	3/20	-1/20	0
0	y_2	1/10	11/5	1	0	-1/60	7/60	0
0	y_6	1/5	24/5	0	0	-1/5	-3/5	1
	$\frac{1}{v}$	1/5	2/5	0	0	2/5	1/5	0

Hence the optimum solution to the given problem is

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/3 & 1/3 & 0 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 1/2 & 1/2 \end{bmatrix} \quad \text{and } v = 1$$

CONCLUSION:

Hence, firm A should adopt strategy A_1 and A_2 with 67 % of time and 33% of time respectively (or with 67% and 43% probability an any one play of the game respectively). Similarly, firm B should adopt strategy B_2& B_3 with 50% of time and 50% of time respectively (or with 50% and 50% probability on any one play of the game respectively).

3.4 EXTENSIONS OF GAMES:

Infact extensive research has been done on a number of more complicated types of games; that is

3.4.1 N-PERSON GAME:

One Such type is N- Person game, where more than two players may participate in the game. This generalization is particularly important because in many kinds of competitive situations, there frequently are more than two competitors involved. This may occur for example, in competition among business firms, in international diplomacy, etc. unfortunately, the existing theory for such games is less satisfactory than it is for two-person games.

3.4.2 NON-ZERO-SUM GAME:

Another generalization is the non-zero sum game, where the sum of the payoffs to the players need not be zero (or any other fixed constant). This case reflects the fact that many competitive situations include non-competitive aspects that contribute to the mutual advantage or mutual disadvantage of the players. For example, the advertising strategies of competing companies can affect not only how they will split the market but also the total size of the market for their competing products.

3.4.3 COOPERATIVE GAME & NON-COOPERATIVE GAME:

Because mutual gain is possible, non zero sum game are further classified in terms of the degree to which the players are permitted to cooperate. At one extreme is the non co-operative game, where there is no pre-play communication between the players. At the other extreme is the cooperative game, where pre-play discussions and binding agreements are permitted. For example competitive situations involving trade regulations between countries, or collective bargaining between labour and management, might be formulated as cooperative games when there more than two players. Cooperative games also allow some or all of the players to from conditions.

3.4.4 INFINITE GAMES:

Still another extension is infinite games. Where the players have an infinite number of pure strategies available to them. These games are designed for the kind of situation where the strategy to be selected can be represented by a continuous decision variable. For example, this decision variable might be the time at which to take a certain action, or the proportion of one's resources to allocate to a certain activity, in a competitive situation.

3.4.5 MARKET GAMES:

One of the earliest applications of game theory was in mathematical economics to describe the bargaining involved in trading. The simple such model suggested by Edgeworth (1981).

3.4.6 META GAMES:

As researchers started to experiment by playing games it become apparent that the outcome actually arrived at were not necessarily those predicted by the theory. Nigel Howard (1966 a, 1966 b, 1971) was interested in why the actual outcomes of non co-operative games like Prisoner's Dilemma might vary from the equilibrium pairs. Howard assumes that each player tries to predict which strategies his opponents will choose in order to plan his own. This leads to idea of an actual stable outcome, where each player correctly predicts all the other player's strategies, and so the outcome. Howard identified such outcomes. Howard identified such outcomes as equilibria of games based on the actual game being played, and called these enlarged games metagames.

3.4.7 MULTI STAGE GAMES:

A multi-stage game is one is which the out come can be a real payoff and a requirement to play the game or another game again. So for the ga,es we have looked at can all be considered as "one-off" games in the sense that there is no obligation to play another game after them. However, when we discussed mixed strategies we mentioned explicity the idea of repeating the game and the concept of repeatedly seeking prices for the product is implicit in the ideas of oligopoly theory.

3.4.8 EVOLUTIONARY GAMES:

One of the most unusual applications of game theory in recent years has been to model how animal behaviour evolves from generation to generation. The way that most genetic characteristics evolve has long been modeled satisfactory using Mendal's principle, e.g. eye colouring in humans. However, there are some characteristics whose evolution cannot be analyzed so simply, because these characteristics affect the reproducing capacity of the individual. The evolution of theses characteristics depends on how individuals with such characteristics interact with individuals with other characteristics.

3.5 METHODS OF SOLUTIONS:

Von Neumann and Morgenstern (1947) introduced the basics of game theory, they also developed the idea of the utility of an out come of the game. In detail about utility theory and the axioms was discussed by Raiff (1968).

" A finite two-person zero-sum game with perfect information has a solution in which both players have pure optimal strategies". This was essentially proved by Zerineto (1913) before the advent of game theory in its present form. The first proof in that context was given by Von Neumann and Morgenstern (1947) and depends on working back through the extensive form of the game. After discovered the simplex method algorithm (Dantzig-1951), as well as Gale, Kuhn and Tucker (1951), pointed out that solving a two – person zero-sum game is equivalent to solving a linear programming was discussed by Jaries (1980).

More modern mathematical treatments about game theory are given by Owen (1968), Vorob'ev (1977), Jones (1980) and Shubik (1982). For finding the conditions under which the minimax theorem holds for infinite games, was discussed by Parathsarathy and Regharan (1971), as well as application of such games both in games of timing, like the continuous time version of the dual games and in mathematical economics was introduced by Aubin (1979) The proof of theorem based on two-person cooperative game by Jones (1980). The theory of cooperative games is even more interesting because it involves some of the most important conflict situation that can arise, Rapoport and Guyer (1966) describe 78 different types of game. For an assuming tour around the application of other cooperative games, Ham burgers's book games as models of social phenomena (1979) is a best book.

Harsanyi (1963) was introduced, generalizes the threat bargaining model to n-person game. The book by Rapoport (1970) is also an easy introductory guide to the main concepts about N-persons games Shubik (1982) gives a comprehensive account of the N-person games. The area of cooperatives n-person games is still the most active area of research in game theory. Haranyi (1959), and in his book (1977) he looks at the whole problem in more detail.

The market games were introduced by Edgeworth (1981). However, the reinterpretation of market problems as games in the sense we understand them, was made by Shubik (1959), and also to re-interpret results from game theory in economic form by Shaptey & Shubik (1969).

Metagame theory was started by Howard in a series of papers in General system (1966a, 1966b, 1970) and described in his book Paradoxes of Rationality (1971). The transformation of metagames by Howard, was continued by Radford, in his books Managerial Decision Making (1975) and complex Decision Problems (1977).

An ever more recent development along the lines of metagame analysis is the "methodique" of hypergame analysis introduced by Bennet (Bennet 1977); Bennet & Huxham (1982). The first paper to isolate the idea of multistage games was Shapley's work (1953) on Stochastic games. Algorithms for solving such games are discussed in Vander Wal (1980)

Evolutionary games are the recent application of game theory the idea of an evolutionary stable strategy was introduced in Maynard –Smith and Price (1973) and Maynard – Smith (1974). The basic mathematical properties of such strategies were worked out by Haigh (1975) and Bishop and Cannings (1978). With Abakuks (1980) printing out an error in their results.

Maynard-Smith (1978) raised the problem of games played between relatives. Grafen (1979) describes how this modification should be made by looking at gene frequency. The evolutionary games with two type of players were introduced by Taylor (1979) and the parental strategy game by Schuster and Sigmund (1981).

3.6 DISCUSSION:

The general problem of how to make decisions in a competitive environment is a very common and important one. The fundamental contribution of game theory is that it provides a basic conceptual framework for formulating and analyzing such problems in simple situations. However, there is a considerable gap between what the theory can handle and the complexity of most competitive situations arising in practice. Therefore, the conceptual tools of game theory usually play just a supplementary role in dealing with these situations.

Because of the importance of the general problem, research is continuing with some success to extend the theory to more complex situations.

Chapter-4

CARGO LOADING PROBLEM

4.1- INTRODUCTION:

The cargo-lading problem deals with the problem of loading items on a vessel with limited volume or weight capacity. Each item produces a level of revenue. The objective is to load the vessel with the most valuable Cargo.

The cargo loading problem is also known as the *fly-away kit* problem, in which a jet pilot must determine the most valuable (emergency) items to take aboard the jet; and also known as *knapsack* problem, in which a soldier (or a hiker) must decide on the most valuable items to carry in a back-pack. It appears that the three names were chosen to ensure equal representation of the navy, air force and army.

Early papers which discuss the knapsack problem in context of dynamic programming include Bellman and Dantzig. Gilmore and Gomary (1961), first utilized Dantzig's dynamic programming approach solving the knapsack problem. A still superior dyanamic programming model was introduced by Gilmore and Gomory (1966).

Enumerative algorithm specialized to solve the knapsack proble include the branch & bound procedure for the zero-one problem appearing in Kolesar (1967). Green berg – Hegerich (1970) procedure is a branch and bound method that utilizes the land-Doig algorithm in an almost straight forward manner.

The Lagrangian multipliers technique was introduce the concept of utilizing to solve discrete programming problems by Everett. Application of the Lagrangian approach to the knapsack problem is described by Fox & Landy and by Gulley, Swanson & Woolsey. Shapiro and Wagner solve the knapsack problem by a network approach. A classical application, which is a knapsack problem is the Lorie and Savage-Capital budgeting model. The use of Balas duality in integer programming to give economic interpretations and properties of optimal solutions to capital budgeting models is discussed by Radha Krishnan and Unger.

Other applications involving the knapsack problem or its solution include the cutting stock problem extensively discussed by Gilmore and Gomory. Using a basic number theoretic result appearing in Mathew's, it can be shown that an integer program with any finite number of constraints can be transformed to a knapsack problem. Bari A. and Ahmad Q.S. (1998) developed conversion of an assignment problem into knapsack problem and its solution procedure. C.E. Ferreira, A. Martin & R. Weismantel (1996) developed a cutting plane based algorithm for the multiple knapsack problem. Cutting planes for Mixed – Integer knapsack polyhedra was developed by X.Q. Yan and E.A. Boyd (1998). Details on Cargo-loading problem was discussed by Dreyfus & Law (1977). A recent contribution to cargo-loading problem was due to Gary E. Horne & Telba Z. Irony (1994).

4.2 FORMULATION OF CARGO LOADING PROBLEM:

Let m_i be the number of units of item i in the cargo. The general problem is represented by the following integer LP.

Maximize $z = r_1 m_1 + r_2 m_2 + ... + r_n m_n$

Subject to,

$$w_1 m_1 + w_2 m_2 + ... + w_n m_n \le W$$

and $$m_{1,} m_{2,} ... , m_n \ge 0 \text{ \& integer}$$

or, Maximize. Z = $\sum_{i=1}^{n} r_i m_i$

Subject to, $\sum_{i=1}^{n} w_i m_i \le W$

and $m_i \ge 0$ and integer; i = 1, 2, ……,n

Where, r_i is the level of revenue of i^{th} unit.

w_i is the unit weight of i^{th} unit.

W is the weight capacity.

4.3.1 METHOD OF SOLUTION BY DYNAMIC PROGRAMMING APPROACH:

The (backward) recursive equation is developed for the general problem of n-item W-ton vessel, i.e.

Or, Maximize $Z = \sum_{i=1}^{n} r_i m_i$

Subject to, $\sum_{i=1}^{n} w_i m_i \leq W$

and $m_i \geq 0$ and integer; i = 1, 2, ……,n

The three elements of the model are:

1- Stage i is represented by item i, i =1, 2, ……,n

2- The alternatives at stage i are represented by m_i, the number of units of item i included in the cargo. The associated return is $r_i m_i$.

Defining $\left[\frac{W}{w_i}\right]$ as the target integer less than or equal to $\frac{W}{w_i}$; it follows that

$m_i = 0, 1, \ldots\ldots\ldots \left[\frac{W}{w_i}\right]$

3- The state at stage i is represented by x_i, the total weight assigned to stages (items) i, i+1, ……, and n, combined. This definition reflects the fact that the weight constraint is the only restriction that links all n stages together.

RECURSIVE EQUATION APPROACH:

Define, $f_i(x_i)$ = maximum return for stages i, i + 1,…… and n, given state x_i

The simplest way to determine recursive equation is a two-step procedure.

Step 1. Express $f_i(x_i)$ as a function of $f_i(x_{i+1})$ as follows:

$$f_i(x_i) = \max_{\substack{m_i = 0,1,\ldots,\left[\frac{W}{w_i}\right] \\ x_i = 0,1,\ldots\ldots\ldots W}} \{r_i m_i + f_{i+1}(x_i + 1)\} \; ; \quad \text{Where, } f_{n+1}(x_{n+1}) \equiv 0$$

Step 2:

Express x_{i+1} as a function x_i to ensure that the left hand side, $f_i(x_i)$, is a function of x_i only. By definition, $x_i - x_{i+1}$ represents the weight consumed at stage i

That is, $x_i - x_{i+1} = w_i m_i$ Or, $x_{i+1} = x_i - w_i m_i$

Thus the proper recursive equation is given as:

$$f_i(x_i) = \max_{\substack{m_i = 0,1,\ldots,\left[\frac{W}{w_i}\right] \\ x_i = 0,1,\ldots\ldots\ldots W}} \{r_i m_i + f_{i+1}(x_i - w_i m_i)\}; \qquad i = 1, 2, \ldots\ldots, n$$

Note that the maximum feasible value of m_i is given by $\left[\frac{W}{w_i}\right]$. This limit will automatically delete all infeasible alternatives for a given value of the state x_i.

4.3.2 METHOD OF SOLUTION BY BRANCH & BOUND TECHNIQUE:

The cargo loading problem yields the linear problem.

$$\text{Maximize } z = \sum_{i=1}^{n} r_i m_i$$

$$\text{Subject to, } \sum_{i=1}^{n} w_i m_i \leq W$$

$$\& \qquad m_i \geq 0;\ i = 1, 2, \ldots\ldots, n$$

Where, m_n is a slack variable (with $w_n = 1$) and $r_n = 0$, when the constraint is initially and inequality. By defining $x_i = w_i m_i$ ($i = 1, \ldots\ldots, n$), we have, $x_i \geq 0$ since, $r_i \geq 0$ and then integer program becomes

$$\text{Maximize } z = \sum_{j=1}^{n} \left({r_i}/{w_i}\right) x_i$$

$$\text{Subject to, } \sum_{j=1}^{n} x_i = W,$$

$$\& \qquad x_i \geq 0 \text{ \& integer}$$

To simplify the notation, we shall assume that the variable have been recorded so that

$\frac{r_1}{w_1} \geq \frac{r_2}{w_2} \geq ....\frac{r_n}{w_n}$, then the linear programming solution to the cargo loading problem is

$m_1 = \frac{W}{w_1}, m_i = 0$, ($i \neq 1$) and the objective function is $r_1 \left[\frac{W}{w_i}\right]$

We can extend this result to the case where the variables are bounded.

The linear program is equivalent to

$$\text{Maximize } z = \sum_{j=1}^{n} \left(r_i \big/ w_i \right) x_i$$

$$\text{Subject to, } \quad \sum_{j=1}^{n} x_i = W$$

$$\& \qquad x_i \geq 0 \text{ \& integer} \qquad (i = 1, 2, \ldots, n)$$

Where, $x_i = w_i m_i$ $\quad (i = 1, 2, \ldots\ldots n)$

If upper integral bounds u_i exist, $m_i \leq u_i$ that $x_i \leq w_i u_i$, since $m_i = x_i \big/ w_i$

In this case the optimal LP solution is $x_1 = W$, $x_i = 0$ ($i = 2, \ldots\ldots, n$) when $w_1 u_1 > W$.

If $w_1 u_1 \leq W$, the optimal LP solution is found by setting $x_i = w_i u_i$ ($i = 1, \ldots, t \geq 1$)

$$X_{t+1} = W - \sum_{i=1}^{t} w_i u_i, \text{ and } x_i = 0 \; (i = t+2, \ldots\ldots n).$$

In terms of the x_i variables, the optimal LP solutions

$$m = \left(\underbrace{\frac{W}{w_1}}_{m_1}, 0, 0 \ldots\ldots.. 0 \right)$$

when $w_1 u_1 > W$ otherwise it is

$$m = \left(u_1, \ldots\ldots, u_t, \underbrace{\left(\frac{1}{w_{t+1}}\right)\left(W - \sum_{i=1}^{t} w_i u_i\right)}_{m_{t+1}}, 0, 0 \ldots.., 0 \right)$$

when, $w_1 u_1 \leq W$, where t is the smallest index,

$$\text{Such that, } W - \sum_{i=1}^{t+1} w_i u_i < 0$$

If positive integral lower bounds u_i exist, then an inequality of the form $m_i \geq u_i$ can be treated implicitly by introducing the complementing variable $\bar{m}_i = m_i - u_i \geq 0$, substituting $u_i + \bar{m}_i$ for m_i in the cargo loading constraint and objective function. This simply reduces W by $m_i u_i$, if $W - m_i u_i < 0$, the problem has no solution, since $m_i > 0$ ($i = 1, \ldots..n$). Otherwise, we have a standard cargo loading problem and associated linear program.

We have shown that a cargo-loading problem with or without bounded variables can be solved as a linear program by inspection. Thus a procedure for solving the integer program is by branch & bound enumeration. If we adopt the Dakin branch & bound procedure, the value of

$m_1 = \frac{W}{w_1}$ is initially examined. If it is integer, the cargo-loading problem has been solved. Otherwise two nodes are created – the first by introducing the constraint $m_1 \leq \left[\frac{W}{w_1}\right]$ and the second by imposing the constraint,

$m_1 \geq \left[\frac{W}{w_1}\right] + 1$ the linear program at each of these nodes is solved by inspection, and the process continues that is nodes are created whenever they may produce an improved integer solution; for example, when the LP solution is not integer and its value exceeds the value for the current best integer solution. As $m_1 = \left[\frac{W}{w_1}\right]$, $m_i = 0$ $(i \neq 1)$ is an integer solution and an initial lower bound on the objective function is $r_1 \left[\frac{W}{w_1}\right]$. Also, observe that if the LP solution is not integer there is exactly one variable (namely x_1 or x_{t+1}), which is fractional, and thus it is selected to create nodes.

4.4 THE PROBLEM AND THE SOLUTION:

A 5 – ton vessel is loaded with one or more of three items, the following table gives the unit weight, w_i, in tons and the unit revenue, r_i, in thousands of dollars for item i. How should the vessel be loaded to maximize the total return?

Item i	w_i	r_i
1	2	65
2	3	80
3	1	30

FORMULATION OF THE PROBLEM:

Let m_i be the number of units of item i (=1, 2, 3) in the cargo. The problem is represented by the following integer linear programming

Maximize $z = 65m_1 + 80m_2 + 30m_3$

Subject to, $2m_1 + 3m_2 + m_1 \le 5$

and $m_1, m_2, m_3 \ge 0$ & integer.

4.4.1 SOLUTION BY DYNAMIC PROGRAMMING PERCURSSIVE EQUATION APPROACH:

Because the unit weight w_i and the maximum weight W all assume integer values the state x_i can assume integer values only.

Stage 3:

The exact weight to be allocated to stage 3 (item 3) is not known in advance but must assume one of the values 0, 1, ..., and 5 (because W = 5 tons). The states $x_3 = 0$ and $x_3 = 5$ respectively, represent the extreme cases of not shipping item 3 at all and of allocating the entire vessel to it. The remaining values of x_3 = (1, 2, 3 and 4) imply a partial allocation of the vessel capacity to item 3. In effect, the given range of values for x_3 covers all the possible allocation of the vessel capacity to item 3.

Because $w_3 = 1$ ton per unit, the maximum number of units of item 3 that can be loaded is $\left[\frac{5}{1}\right] = 5$, which means that the possible values of m_3 are 0,1,2,3,4, and 5. An alternative m_3 is feasible only if $w_3 m_3 \leq x_3$. Thus, all the infeasible alternatives (those for which $w_3 m_3 \leq x_3$) are excluded. The following equation is the basis for comparing the alternatives of stage 3.

$$f_3(x_3) = \max_{m_3}\{30m_3\}; \quad m_3 = \left[\frac{5}{1}\right] = 5$$

The following tableau compares the feasible alternatives for each value of x_3.

	30 m_3						Optimal solution	
x_3	$m_3=0$	$m=_1$	$m=_2$	$m=_3$	$m=_4$	$m=_5$	$f_3(x_3)$	m_3*
0	0	--	--	--	--	--	0	0
1	0	30	--	--	--	--	30	1
2	0	30	60	--	--	--	60	2
3	0	30	60	90	--	--	90	3
4	0	30	60	90	120	--	120	4
5	0	30	60	90	120	150	150	5

Stage2:

$$f_2(x_2) = \max_{m_2}\{80m_2 + f_3(x_3 - 3m_2)\}$$

$$\max m_2 = \left[\frac{5}{3}\right] = 1$$

80 m_2+f_3 (x_2−3m_2)			**Optimal Solution**	
X_2	$m_{2=0}$	$m_{2=1}$	$f_2(x_2)$	m_2*
0	0 + 0 =0	--	0	0
1	0 + 30 = 30	--	30	0
2	0 + 60 = 60	--	60	0
3	0 + 90 =90	80 + 0 = 80	90	0
4	0 + 120 = 120	80 + 30 = 110	120	0
5	0 + 150 = 150	80 + 60 = 140	150	0

Stage 1:

$$f_1(x_1) = \underset{m_1}{Max}\{65m_1 + f_2(x_1 - 2m_1)\}$$

$$\text{max. } m_1 = \left[\frac{5}{3}\right] = 2$$

	$65m_1 + f_2(x_1 - 2m_1)$			Optimal solution	
x_1	$m_{1=0}$	$m_{1=1}$	$m_{1=2}$	$f_1(x_1)$	m_{1*}
0	0 + 0 =0	--	--	0	0
1	0 + 30 = 30	--	--	30	0
2	0 + 60 = 60	65 + 0 = 65	---	65	1
3	0 + 90 =90	65 + 30 = 95	--	95	1
4	0 + 120 = 120	65 + 60 =125	130 + 0 = 130	130	2
5	0 + 150 = 150	65 + 95 = 155	130 + 30 = 160	160	2

The optimal solution is now determined in the following manner:

Given W = 5 tons, from stage 1, $x_{1=5}$ gives the optimum alternative $m_{1*=2}$, which means that 2 unit of item 1 will be loaded on the vessel. This allocation leaves.$x_2 = x_1 - 2m_2^* = 5 - 2 \times 2 = 1$, from stage 2, $x_{2=1}$ yields $m_2^* = 0$, which in, turn, gives $x_3 = x_2 - 3m_2^* = 1 - 3 \times 0 = 1$, from stage 3, $x_{3=1}$ gives $m_{1*} = 1$,

Thus the complete optimal solution is $m_1^* = 1$, $m_2^* = 0$, $m_3^* = 2$. The associated return is $ 160,000.

Note: In the table for stage 1, we actually need to obtain the optimum for $x_1 = 5$ only, because this is the last stage to be considered. However, the computations for $x_1 = 0, 1, 2, 3$, and 4 are included to allow carrying out sensitivity analysis. For example, what would happen if the vessel capacity is 4 tons in place of 5 tons? The new optimum solution can be determined by starting with $x_{1=4}$ at stage 1 and continuing in the same manner as we did for $x_{1=5}$

4.4.2 SOLUTION BY BRANCH & BOUND TECHNIQUE:

Maximize $Z = 65\, m_1 + 80\, m_2 + 30\, m_3$

Subject to, $2m_1 + 3m_2 + m_1 \leq 5$

And $m_i \geq 0$ & integer; i=1, 2, 3,

The computations are tabulated in table 4.1 and the associated tree appears in figure 4.0. We are using the node selection rule: "choose the most recently create node with the highest linear programming solution to create nodes from".

The integer solution $m_1 = \left[\frac{5}{2}\right] = 2$, $m_2 = m_3 = 0$ gives the initial lower bound $65 \times 2 = 130$.

The nodes in table 4.0 and fig. 4.1 are numbered in order of their creation. An active node naturally is one that can possibly produce an improved integer solution.

Calculation:

At node (2) We have the additional constraints $m_1 \leq 2$ letting, $2 - m_1 = \bar{m}_1 \geq 0$ and substituting $2 - \bar{m}_1$ for m_1 in the cargo loading constraints yields.

$$2\left(2 - \bar{m}_1\right) + 3m_2 + m_3 \leq 5 \quad \text{or} \quad -2\bar{m}_1 + 3m_2 + m_3 \leq 1 \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

As $3(1) > 1$, the LP solution is, $m_2 = 1/3$, $\bar{m}_1 = 0$ or $m_1 = 2$, $m_3 = 0$

m_1	m_2	m_3	Value
2	1/3	0	$156\frac{2}{3}$

----------node (2)

At node 3,

$m_1 \geq 3$, i.e. $\bar{m}_1 = m_1 - 3 \geq 0$, $\Rightarrow m_1 = 3 + \bar{m}_1$, yields,

$$2\left(3 + \bar{m}_1\right) + 3m_2 + m_3 \leq 5 \quad \text{or} \quad 2\bar{m}_1 + 3m_2 + m_3 \leq -1 \ldots\ldots\ldots\ldots\ldots (ii)$$

The LP solution is infeasible at node (3)

At node 4,

$m_2 \geq 1$, Letting $m_2 - 1 = \bar{m}_2 \Rightarrow m_2 = 1 + \bar{m}_2$, yields

$$2m_1 + 3\left(1 + \bar{m}_2\right) + m_3 \leq 5 \quad \text{or} \quad 2m_1 + 3\bar{m}_2 + m_3 \leq 2 \ldots\ldots\ldots (iii)$$

$m_1 = 1$, $\bar{m}_2 = 0 \Rightarrow m_2 = 1$, $m_3 = 0$

m_1	m_2	m_3	Value
1	1	0	145

---------node (4)

At node (5),

$m_2 = 0$

from (1) $\Rightarrow m_3 = 1$, $\bar{m}_1 = 0 \Rightarrow m_1 = 2$

m_1	m_2	m_3	Value
2	0	1	160

---------node (5)

At node (6),

$m_1 \leq 1$, i.e., $1 - m = \bar{m}_1 \Rightarrow m_1 = 1 - \bar{m}_1$, yields

$2(2 - \bar{m}_1) + 3m_2 + m_3 \leq 5$--------(iv)

m_1	m_2	m_3	Value
1	1	0	145

---------node (6)

At node 8,

$m_3 = 0$, from (1) $m_2 = \frac{1}{3}$, $\bar{m}_1 = 0 \Rightarrow m_1 = 2$

m_1	m_2	m_3	Value
2	$\frac{1}{3}$	0	$156\frac{2}{3}$

---------node (8)

At node 9,

$m_3 \geq 1$, $m_3 - 1 = \bar{m}_3 \Rightarrow m_3 = 1 + \bar{m}_3$, yields

$2m_1 + 3m_2 + \bar{m}_3 \leq 4$---------- (v)

$m_1 = 2$, $\bar{m}_3 = 0 \Rightarrow m_3 = 1$, $m_2 = 0$

m_1	m_2	m_3	Value
2	0	1	160

----------node (9)

At node 10, The LP Solution is not an optimal integer solution ($\because m_1 = 2$, $m_2 = 1$, $m_3 = 0$ are does not satisfied the cargo loading constraints).

At node 11, the LP solution is infeasible.

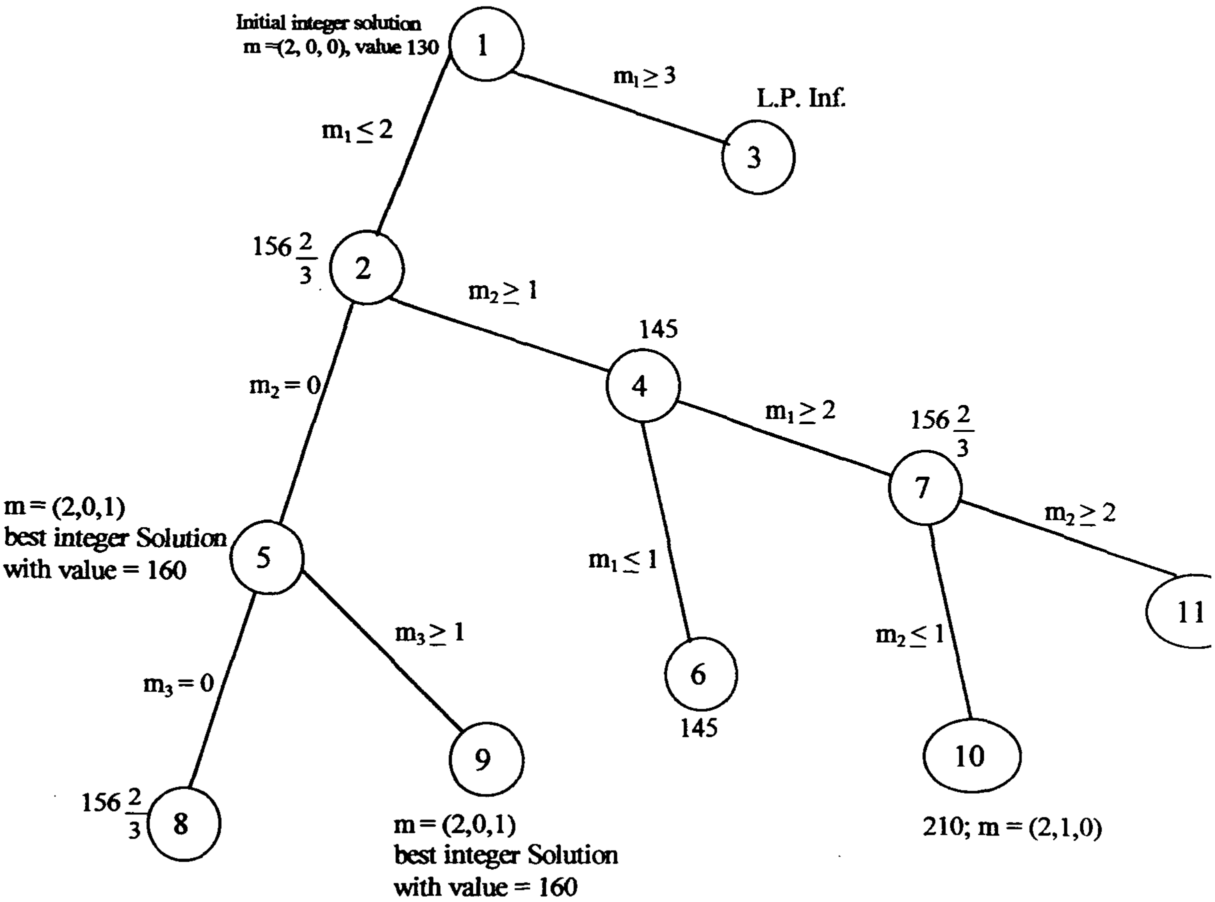

Figure 4.0

Table 4.1

Nodes	Constraints active at node			L.P. Solution				Value of the best integer solution	Active nodes selected	Remarks
				Variable			Value			
	m_1	m_2	m_3	m_1	m_2	m_3				
1	-	-	-	5/2	0	0	162 1\2	130	(1)	130 from, $r_1 m_1 = 65$ [5/2]
2	≤ 2	-	-	2	1/3	0	156 2/3	130	(1), 2	
3	≥ 3	-	-	Infeasible					(2)	
4	≤ 2	≥ 1	-	1	1	0	145	145	(2), 4	
5	≤ 1	$= 0$	-	2	0	1	160	160	(2), 4 ,5	Optimal integer solution
6	$= 2$	≥ 2	-	1	1	0	145	145	4, (5), 7	$m_1 \leq 2, m_1 \leq 1, \Rightarrow m_1 \leq 1$
7	≤ 2	≥ 1	-	2	1/3	0	156 2/3	130	(5), 7	$m_1 \leq 2, m_1 \geq 2, \Rightarrow m_1 = 2$
8	≤ 2	$= 0$	$= 0$	2	1/3	0	156 2/3	130	(5)	
9	≤ 2	$= 0$	≥ 1	2	0	1	160	160	None	Optimal integer solution
10	$= 2$	$= 1$	-	2	1	0	210	210	-	Not optimal integer solution
11	≥ 2	$= 2$	-	Infeasible					-	Because; $\left.\begin{array}{l} m_1 \leq 2, m_1 \geq 2, \Rightarrow m_1 = 2 \\ m_2 \geq 1, m_2 \leq 1, \Rightarrow m_2 = 1 \end{array}\right\}$

4.5 DISCUSSION:

The cargo-loading problem represents a typical resource allocation model in which a limited resource is apportioned among a finite number of economic activities. The objective is to maximize an associated return function. In such models, the definition of the state at each stage will be similar to the definition given for the cargo-loading model. Normally, the state at stage i is the total resource amount allocated to stage i, i + 1, ... and n.

Chapter-5

OPTIMUM ALLOCATION IN SATISFIED RANDOM SAMPLING

5.1 INTRODUCTION:

In the development of theory underlying statistical methods, one is often faced with an optimization problem; the fundamental paper by Charnes, Cooper and Ferguson (1955) introduced the application of mathematical programming to statistics. Other areas of application of mathematical programming in statistics developed simultaneously. Kokan and Khan (1967), Bruvold and Murphy (1978), and Rao (1979) in sampling; Vinod (1969), Rao (1971) in cluster analysis; Foody and Hedayat (1977) in the construction of BIB design with repeated blocks; Dantzig and Wald (1951), Krafft (1970) and Pulkelshein (1978) in testing statistical hypothesis. Optimum allocation in stratified sampling has been considered by Dalenius (1957), Folks and Antle (1965), Gosh (1965) and Kokan and Khan (1967), among others. Khan S.U. and Bari A. (1977) Developed a procedure for integer solution to some allocation problems.. In this chapter numerical example taken from cohran (1977).

Sampling theory deals with problems associated with selection of samples from a population according to certain probability mechanisms. In sampling theory, stratified random sampling occupies and important place. In stratified sampling, the population of N units is first divided into sub populations of N_1, N_2 .....,N_L units, respectively called strata. Population characteristics can be inferred with samples from each stratum, exploiting the gain in precision in the estimates, administrative convenience, and the flexibility of using different sampling procedure in the different sub populations.

5.2 THE PROBLEM:

Let, N_i be the number of units in the i^{th} stratum and $\sum_{i=1}^{L} N_i = N$, where L is the number of strata into which the N units are divided. Let n_i be the size of sample drawn from the i^{th} stratum. Assume that the samples are drawn independently in different strata.

The problem of optimally choosing the n_i's is known as the "optimal allocation problem". The objective in this problem might be minimization of the variance of the estimate of the population characteristics under study, with restriction on the total number of samples drawn or on the total budget available.

Define:

$$\bar{y}_i = \frac{1}{n_i}\sum_{h=1}^{n_i} y_{ih} \text{ sample mean}$$

i.e. $\bar{y}_i$ is an unbiased estimate of stratum mean $\bar{Y}_i$

$$\bar{y}_{st} = \frac{1}{N}\sum_{i=1}^{L} N_i \bar{y}_i = \text{estimate of the population mean } \bar{Y}$$

i.e $\bar{y}_{st}$ is an unbiased estimate of the population mean $\bar{Y}$

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{L} \bar{y}_i N_i = \frac{1}{n_i}\sum_{i=1}^{L}\sum_{h=1}^{N_i} y_{ih} = \bar{Y}$$

Where, y_{ih} is the value of y for the h[th] unit in the i[th] stratum.

As the precision of this estimate is measured by variance of the sample estimate by the definition

$$(\bar{y}_{st} - \bar{Y})^2 = \left[\sum_{i=1}^{L} N_i \left(\frac{\bar{y}_i - \bar{Y}_i}{N}\right)\right]$$

$$= \frac{1}{N^2}\sum^{L} N_i^2 \left(\bar{y}_i - \bar{Y}_i\right)^2 + \frac{2}{N^2}\sum_{i \geq i'} N_i N_{i'} \left(\bar{y}_i - \bar{Y}_i\right)^2 \left(\bar{y}_{i'} - \bar{Y}_{i'}\right)^2$$

Averaging over all sample and noticing the fact that the cross product term vanish, we get,

$$\mathrm{V}(\bar{y}_{st}) = \frac{1}{N^2}\sum_{i=1}^{L} N_i^2 E \left(\bar{y}_i - \bar{Y}_i\right)^2$$

$$= \frac{1}{N^2}\sum_{i=1}^{L} N_i^2 \ \mathrm{V}(\bar{y}_i)$$

However, $\mathrm{V}(\bar{y}_i)$ has the expression

$$V(\bar{y}_i) = S_i^2 \frac{(N_i - n_i)}{N_i n_i} = S_i^2 \left(\frac{1}{n_i} - \frac{1}{N_i}\right)$$

Where, $S_i^2 = \frac{1}{N_i - 1}\sum_{h=1}^{N_i}\left(y_{ih} - \bar{y}_i\right)^2$ population mean square.

Let, $W_i = \frac{N_i}{N}$ = stratum weight. And, $x_i = \frac{1}{n_i} - \frac{1}{N_i}$, then we have,

$$V(\bar{y}_{st}) = \sum_{i=1}^{L} W_i^2 S_i^2 x_i$$

5.3 FORMULATION OF THE PROBLEM AS AN INTEGER NONLINEAR PROGRAMMING PROBLEM:

We consider the problem of choosing n_i, i =1,2 ……., L such that the sum of these n_i equals n a fixed total sample size, and the $v(\bar{y}_{st})$ is a minimum, this problem can be formulated as:

$$\left.\begin{aligned} &\text{Minimize } \sum_{i=1}^{L} W_i^2 S_i^2 x_i \\ &\text{Subject to, } \sum_{i=1}^{L} n_i = n \\ &1 \le n_i \le N_i, \; n_i \text{ integer}, i = 1, \ldots\ldots, L \end{aligned}\right\} \text{--------(5.2.1)}$$

5.3.1 LAGRANGIAN MULTIPLIER METHOD FOR FINDING OPTIMAL (n_i):

Let, $a_i = W_i^2 S_i^2$; i =1,……,L, then the objective function.

$$\sum_{i=1}^{L} W_i^2 S_i^2 x_i = \sum_{i=1}^{L} a_i x_i$$

$$\sum_{i=1}^{L} a_i\left(\frac{1}{n_i} - \frac{1}{N_i}\right) = \sum_{i=1}^{L}\frac{a_i'}{n_i} - \sum_{i=1}^{L}\frac{a_i'}{N_i}$$

But $\sum_{i=1}^{L}\frac{a_i}{N_i}$ is a constant, therefore it is sufficient to consider minimizing $\sum_{i=1}^{L}\frac{a_i'}{n_i}$

i.e.

$$\left.\begin{aligned} &\text{Minimize } \sum_{i=1}^{L}\frac{a_i}{n_i} \\ &\text{Subject to, } \sum_{i=1}^{L} n_i = n \\ &1 \le n_i \le N_i, n_i \text{ integer; } i = 1, \ldots\ldots L \end{aligned}\right\} \text{---------(5.2.2)}$$

If the restrictions that n_i must be a positive integer and bounded above by N_i, $\forall i$ are relaxed. Then classical lagrangian multiplier method can be used to find optimal n_i, we have

$$n_i = n \frac{\sqrt{a_i}}{\sum_{i=i}^{L}\sqrt{a_i}} \text{-------------(5.2.3)}$$

However, there are three eventualities:

1) $n_i > N_i$ for some i, in sampling literature eventuality (1) is referred to as over sampling – i.e. the optimal allocation requires sampling more than 100% in certain strata. Non-integer solution are rounded off.
2) n_i may not be an integer for every i.
3) $n_i < 1$, for some i, in this we do not have a solution and it can be easily taken care of by assuming that we sample at least one unit from each stratum, and allocating the rest of n-L units optimally.

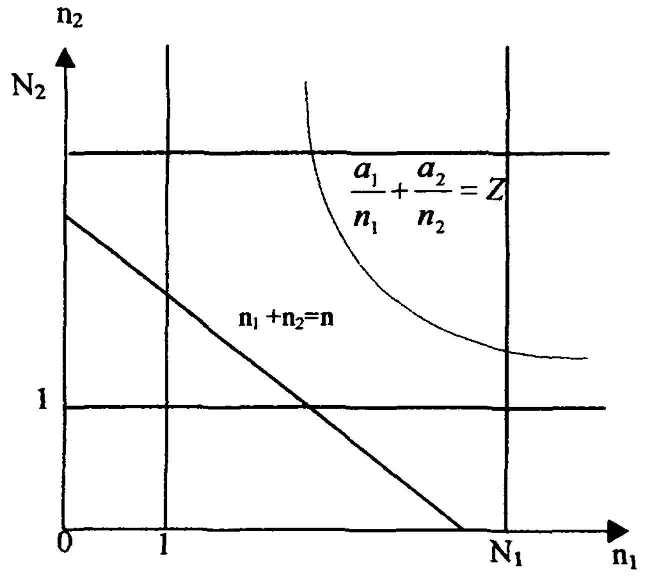

Figure 5.1 Feasible region & objective function when N_1 & N_2 are both larger than n.

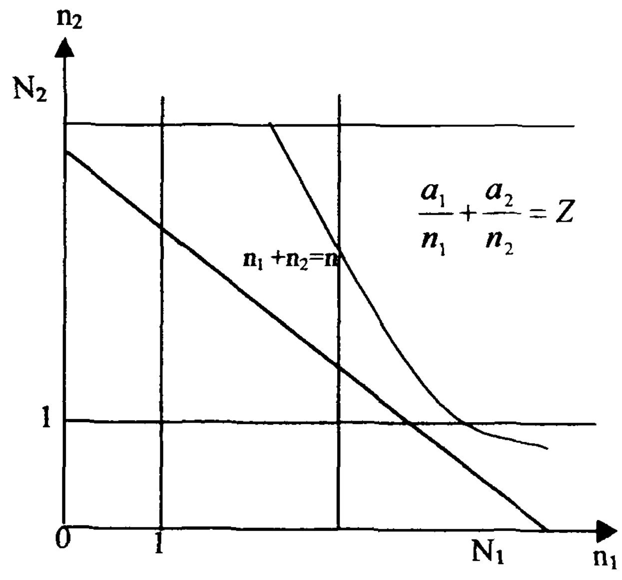

Figure 5.2 Feasible region & objective function wher only N_2 is larger than n.

Notice that $\frac{1}{n_i}$ is strictly convex in each i we find the objective function to be a strictly convex function if $a_i > 0$, that is, $S_i^2 > 0$, $\forall i$, then we are interested in minimizing a strictly convex function over a bounded convex region, created by a linear equality and 2L upper and lower bound restrictions

When L = 2, the feasible region and the objective function appears as in figure 5.1. In fig. 5.1 both N_1 & N_2 are larger tan n other wise, we may have the configuration shown in fig. 5.2.

5.3.2. DETERMINING OPTIMAL ALLOCATION (Integer n_i) BY DYNAMIC PROGRAMMING RECURSIVE APPROACH:

We have a non – linear objective function and a linear equality restriction,

$$n_1 + \ldots\ldots + n_L = n,$$

and upper and lower bound restrictions on n_i $1 \le n_i \le N_i$, i = 1,.......,L and each n_i is an integer.

Let us define f (k, r) to be the minimal value of the objective function, using only the first k strata with total sample size r. That is,

$$\left.\begin{aligned} & f(k, r) = \min \sum_{i=1}^{k} \frac{a_i}{n_i} \\ & \text{Subject to, } \sum_{i=1}^{k} n_i = r \\ & \qquad \text{And } \quad i \le n_i \le N_i, n_i \text{ integer}; \quad i = 1, \text{----}, k \end{aligned}\right\} \text{--------(5.2.4)}$$

Thus, the problem (5.2.1) is equivalent to the problem of finding f (L, n). f (L, n) is found recursively, by finding f (k, r) for k = 1, ...,L and r = 0, 1, ...,n.

$$\text{Now,} \quad f(k, r) = \min \left[\frac{a_k}{n_k} + \sum_{i=1}^{k-1} \frac{a_i}{n_i} \right]; \text{ subject to, } \sum_{i=1}^{k-1} n_i = r - n_k$$

$$\& \ 1 \le n_i \le N_i, \ n_i \text{ integer}; \quad i = 1, \ldots.., k-1$$

For a fixed integer value of n_k , $1 \le n_k \le \min [\, r, N_k]$

i.e. $f(k,r) = \frac{a_k}{n_k} + \left\{ \min \left[\sum_{i=1}^{k-1} \frac{a_i}{n_i} \right] \right\}$

Subject to, $\sum_{i=1}^{k-1} n_i = r - n_k$

& $\quad 1 \leq n_i \leq N_i$, n_i integer; $i = 1, \ldots, k-1$

But by definition, the term in the braces is equal to $f(k-1, r-n_k)$.

Suppose we assume that for a given k, $f(k-1, r)$ is known for all possible $r = 0, 1, \ldots, n$ then

$$f(k, r) = \min_{n_k = 1, \ldots, n} \left[\frac{a_k}{n_k} + f(k-1, r-n_k) \right] \quad n_k = 1, \ldots, n \qquad \text{-----(5.2.5)}$$

This formula is known as the "**Dynamic Programming Recursive Formula**".

NOTE: Using (5.2.5) for each $k = 1, 1, \ldots, L$ and $r = 0, 1, \ldots, n$, $f(L,n)$ can be calculated. Initially we set $f(k, r) = \infty$ if $r < k$ since we wish to have $n_i \geq 1$ for each $i = 1, \ldots, k$, & r must be at least equal to k.

Also,

$f(1, r) = \min \left[\frac{a_1}{n_1} \right]$; Subject to, $n_i = r$; & $1 \leq n_i \leq N_1$

i.e. $f(1, r) = \begin{cases} \infty & \text{for } r > N_1 \text{ or, } r < 1 \\ a_1/r & \text{for } 1 \leq r \leq N_1 \end{cases}$

We tabulate the values of $f(k, r)$ and the optimal n_k, for each k systematically. Then from $f(L, n)$, optimal n_L can be found; from $f(L-1, n-n_L)$ optimal n_{L-1} can be found; and so on, until finally we find optimal n_1.

5.4 An Example

Table 5.4.1 shows the no. of inhabitants, of 64 large cities in the U.S., in thousands, for the year 1930. The cities are grouped into three strata.

Table 5.4.1

Number if inhabitants, in thousands, for the year 1930, in 64 large cities of the U.S. (by stratum)

1			2			3		
90	58	36	31	25	20	14	14	16
82	49	32	27	23	18	17	12	12
78	44	33	28	26	16	15	13	12
81	45	30	25	29	20	14	13	13
67	46	29	27		15	11	10	
124	46	29	21		16	12	11	
57	40	25	26		14	12	11	
63	37	29	21		17	15	11	

There are 16, 20, and 28 cities, respectively, in the first, second and third stratum. We calculate a_i's as follows: first we compute ΣY, $\bar{Y}$, S_i^2 and W_i for each stratum (see table 5.4.2)

Table 5.4.2

Stratum i	N_i	$(\Sigma Y)_i$	$\bar{Y}_i$	S_i^2	W_i
1	16	1007	62.9375	540.0625	0.2500
2	20	552	27.6000	14.6737	0.3125
3	28	394	14.0714	7.2540	0.4375
Total	**64**	**1953**			

Now, a_i is given by $W_i^2 S_i^2$ we find

$a_1 = 33.7539$ $a_2 = 1.4330$ and $a_3 = 1.3885$

Suppose we wish to allot optimally a total of 24 samples among the three strata. We obtain the continuous solution given by

$$n_i^* = n \frac{\sqrt{a_i}}{\sum \sqrt{a_i}}$$

We get $n_1^* = 17.0350$, $n_2^* = 3.5100$, and $n_3^* = 3.4549$. The rounded –off integer solutions are 17, 4 and 3. However, stratum 1 has only 16 cities in all.

Therefore this solution is not feasible – i.e., we have the problem of over sampling. So we resort to the dynamic recursive approach. First we calculate f (1, r), then we calculate

$$f(2, r) = \underset{[n_2\ feasible]}{\min} \left[\frac{a_2}{n_2} + f(1, r - n_2 \right]$$

And note down the optimal n_2 for each r. using f(2, r) we compute f(3, r).

Table 5.4.3 gives f (k, r), for k = 1, 2 and f (3, 24). Thus we find n_3 = 4, and f (3, 24) = 2.8150. With r = 24-4 = 20 and k= 2, we get n_2 = 4. Finally r = 20-4 = 16 and k= 1, we find n_1 = 16.

Therefore 16, 4, 4, turns out to be optimal.

Table 5.4.3

r	f(1,r)	n_1	f(2, r)	n_2	f(3, r)	n_3
0	--	--	--	--		
1	33.7539	1	--	--		
2	16.8769	2	35.1869	1		
3	11.2513	3	18.3099	1		
4	8.4385	4	12.6843	1		
5	6.7508	5	9.8715	1		
6	5.6257	6	8.1838	1		
7	4.8220	7	7.0587	1		
8	4.2192	8	6.2550	1		
9	3.7504	9	5.5385	2		
10	3.3754	10	4.9357	2		
11	3.0685	11	4.4669	2		
12	2.8128	12	4.0919	2		
13	2.5964	13	3.7850	2		
14	2.4110	14	3.5293	2		
15	2.2403	15	3.2905	3		
16	2.1096	16	3.0741	3		
17	--		2.8887	3		
18	--		2.7280	3		
19	--		2.5873	3		
20	--		2.4679	4		
21	--		2.3962	5		
22	--		2.3484	6		
23	--		2.3143	7		
24	--		2.2887	8	2.8150	4

5.5 A VARIATION OF THE PROBLEM:

This approach can be easily extended to the problem of optimally allocating the sample size, subject to budget restriction, instead of the restriction on the total sample size, namely n.

Suppose the cost per sample differs for the different strata. Let c_i be the cost per sample in the i^{th} stratum. Let c be the total budget available. Then we wish to,

$$\left.\begin{aligned} &\text{Minimize } \sum_{i=1}^{L}\frac{a_i}{n_i} \\ &\text{Subject to, } \sum c_i n_i = c \\ &\text{And} \quad 1 \le n_i \le N_i,\ n_i \text{ integer for } i = 1,\ldots\ldots L \end{aligned}\right\}\text{-------(5.2.6)}$$

We have the recursion formula given by

$$f(k, c) = \min_{n_k \text{ feasible}} \left[\frac{a_k}{n_k} + f(k-1, c - c_k n_k)\right]$$

$$\text{Where, } f(k, c) = \text{Min} \sum_{i=1}^{k}\frac{a_i}{n_i}$$

$$\text{Subject to, } \sum c_i n_i \le c$$

$$\& \quad 1 \le n_i \le N_i,\ n_i \text{ integer for } i = 1,\ldots\ldots k$$

For all c feasible – i.e. $\sum_{i=1}^{k} c_i \le c \le C$. The method of finding optimal n_i is exactly as in the (5.5) Example.

5.6 DISCUSSION:

There are various situations in sample surveys, which can be formulated as optimization problems. In some problems we can use the well-known Lagrange multipliers technique. Those problems, which cannot be solved by Lagrange multipliers technique or other methods of calculus, can be formulated as Mathematical programming problems and special type of algorithms can be developed for them. Some problems like allocation of sample numbers to different strata in multivariate surveys are found as convex separable programming problems. A procedure is given for obtaining first a non-integral solution. The Branch and Bound procedure is then for obtaining an integer solution.

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